Chapter 4

The Integral

Integration, defined as a continuous summation process, is linked to differentiation by the fundamental theorem of calculus.

In everyday language, the word *integration* refers to putting things together, while *differentiation* refers to separating, or distinguishing, things.

The simplest kind of "differentiation" in mathematics is subtraction, which tells us the difference between two numbers. We may think of differentiation in calculus as telling the difference between the values of a function at nearby points in its domain.

By analogy, the simplest kind of "integration" in mathematics is addition. Given two or more numbers, we can put them together to obtain their sum. Integration in calculus is an operation on functions, giving a "continuous sum" of all the values of a function on an interval. This process can be applied whenever a physical quantity is built up from another quantity which is spread out over space or time. For example, in this chapter, we shall see that the distance travelled by an object moving on a line is the integral of its velocity with respect to time, generalizing the formula "distance = velocity × time," which is valid when the velocity is constant. Other examples are that the volume of a wire of variable cross-sectional area is obtained by integrating this area over the length of the wire, and the total electrical energy consumed in a house during a day is obtained by integrating the time-varying power consumption over the day.

4.1 Summation

The symbol $\sum_{i=1}^{n} a_i$ *is shorthand for* $a_1 + a_2 + \cdots + a_n$.

To illustrate the basic ideas and properties of integration, we shall reexamine the relationship between distance and velocity. In Section 1.1 we saw that velocity is the time derivative of distance travelled, i.e.,

$$\text{velocity} \approx \frac{\Delta d}{\Delta t} = \frac{\text{change in distance}}{\text{change in time}} . \tag{1}$$

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In this chapter, it will be more useful to look at this relationship in the form

$$\Delta d \approx \text{velocity} \times \Delta t. \tag{2}$$

To be more specific, suppose that a bus is travelling on a straight highway and that its position is described by a function $y = F(t)$, where y is the position of the bus measured in meters from a designated starting position, and t is the time measured in seconds. (See Fig. 4.1.1.) We wish to obtain the

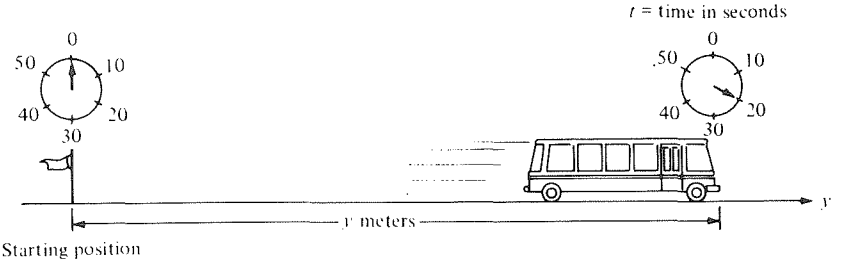

Figure 4.1.1. What is the position of the bus in terms of its velocity?

position y in terms of velocity v. In Section 2.5 we did this by using the formula $v = dy/dt$ and the notion of an antiderivative. This time, we shall go back to basic principles, starting with equation (2).

If the velocity is constant over an interval of length Δt, then the approximation ($\approx$) in equation (2) becomes equality, i.e., $\Delta d = v\,\Delta t$. This suggests another easily understood case: suppose that our time interval is divided into two parts with durations Δt_1 and Δt_2 and that the velocity during these time intervals equals the constants v_1 and v_2, respectively. (This situation is slightly unrealistic, but it is a convenient idealization.) The distance travelled during the first interval is $v_1 \Delta t_1$ and that during the second is $v_2 \Delta t_2$; thus, the total distance travelled is

$$\Delta d = v_1 \Delta t_1 + v_2 \Delta t_2 .$$

Continuing in the same way, we arrive at the following result:

Summation, Distance, and Velocity

If a particle moves with a constant velocity v_1 for a time interval Δt_1, v_2 for a time interval Δt_2, v_3 for a time interval Δt_3, . . . , and velocity v_n for time interval Δt_n, then the total distance travelled is

$$\Delta d = v_1 \Delta t_1 + v_2 \Delta t_2 + v_3 \Delta t_3 + \cdots + v_n \Delta t_n . \tag{3}$$

In (3), the symbol "$+ \cdots +$" is interpreted as "continue summing until the last term $v_n \Delta t_n$ is reached."

Example 1 The bus in Fig. 4.1.1 moves with the following velocities:

4 meters per second for the first 2.5 seconds,
5 meters per second for the second 3 seconds,
3.2 meters per second for the third 2 seconds, and
1.4 meters per second for the fourth 1 second.

How far does the bus travel?

Solution We use formula (3) with $n = 4$ and

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$$v_1 = 4, \qquad \Delta t_1 = 2.5,$$
$$v_2 = 5, \qquad \Delta t_2 = 3,$$
$$v_3 = 3.2, \qquad \Delta t_3 = 2,$$
$$v_4 = 1.4, \qquad \Delta t_4 = 1$$

to get

$$\begin{aligned}\Delta d &= 4 \times 2.5 + 5 \times 3 + 3.2 \times 2 + 1.4 \times 1 \\ &= 10 + 15 + 6.4 + 1.4 \\ &= 32.8 \text{ meters.} \blacktriangle\end{aligned}$$

Integration involves a summation process similar to (3). To prepare for the development of these ideas, we need to develop a systematic notation for summation. This notation is not only useful in the discussion of the integral but will appear again in Chapter 12 on infinite series.

Given n numbers, a_1 through a_n, we denote their sum $a_1 + a_2 + \cdots + a_n$ by

$$\sum_{i=1}^{n} a_i \tag{4}$$

Here Σ is the capital Greek letter *sigma*, the equivalent of the Roman S (for *sum*). We read the expression above as "the sum of a_i, as i runs from 1 to n."

Example 2 (a) Find $\sum_{i=1}^{4} a_i$, if $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 6$. (b) Find $\sum_{i=1}^{4} i^2$.

Solution (a) $\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4 = 2 + 3 + 4 + 6 = 15$.

(b) Here $a_i = i^2$, so

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30. \blacktriangle$$

Notice that formula (3) can be written in summation notation as

$$\Delta d = \sum_{i=1}^{n} v_i \Delta t_i . \tag{3'}$$

The letter i in (4) is called a *dummy index*; we can replace it everywhere by any other letter without changing the value of the expression. For instance,

$$\sum_{k=1}^{n} a_k \quad \text{and} \quad \sum_{i=1}^{n} a_i$$

have the same value, since both are equal to $a_1 + \cdots + a_n$.

A summation need not start at 1; for instance

$$\sum_{i=2}^{6} b_i \quad \text{means} \quad b_2 + b_3 + b_4 + b_5 + b_6$$

and

$$\sum_{j=-2}^{3} c_j \quad \text{means} \quad c_{-2} + c_{-1} + c_0 + c_1 + c_2 + c_3 .$$

Example 3 Find $\sum_{k=2}^{5} (k^2 - k)$.

Solution $\sum_{k=2}^{5} (k^2 - k) = (2^2 - 2) + (3^2 - 3) + (4^2 - 4) + (5^2 - 5) = 2 + 6 + 12 + 20 = 40. \blacktriangle$

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Summation Notation

To evaluate

$$\sum_{i=m}^{n} a_i,$$

where $m \leqslant n$ are integers, and a_i are real numbers, let i take each integer value such that $m \leqslant i \leqslant n$. For each such i, evaluate a_i and add the resulting numbers. (There are $n - m + 1$ of them.)

We list below some general properties of the summation operation:

Properties of Summation

1. $\displaystyle\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i.$

2. $\displaystyle\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i,$ where c is a constant.

3. If $m \leqslant n$ and $n + 1 \leqslant p$, then

$$\sum_{i=m}^{p} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i.$$

4. If $a_i = C$ for all i with $m \leqslant i \leqslant n$, where C is some constant, then

$$\sum_{i=m}^{n} a_i = C(n - m + 1).$$

5. If $a_i \leqslant b_i$ for all i with $m \leqslant i \leqslant n$, then

$$\sum_{i=m}^{n} a_i \leqslant \sum_{i=m}^{n} b_i.$$

These are just basic properties of addition extended to sums of many numbers at a time. For instance, property 3 says that $a_m + a_{m+1} + \cdots + a_p = (a_m + \cdots + a_n) + (a_{n+1} + \cdots + a_p)$, which is a generalization of the associative law. Property 2 is a distributive law; property 1 is a commutative law. Property 4 says that repeated addition of the same number is the same as multiplication; property 5 is a generalization of the basic law of inequalities: if $a \leqslant b$ and $c \leqslant d$, then $a + c \leqslant b + d$.

A useful formula gives the sum of the first n integers:

Sum of the First n Integers

$$\sum_{i=1}^{n} i = \frac{1}{2} n(n + 1) \tag{5}$$

To prove this formula, let $S = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n$. Then write S again with the order of the terms reversed and add the two sums:

$$\begin{array}{l} S = 1 \quad + 2 \quad + 3 \quad + \cdots + (n-2) + (n-1) + n \\ S = n \quad + (n-1) + (n-2) + \cdots + 3 \quad + 2 \quad + 1 \\ \hline 2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \end{array}$$

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Since there are n terms in the sum, the right-hand side is $n(n+1)$, so $2S = n(n+1)$, and $S = \frac{1}{2}n(n+1)$.

Example 4 Find the sum of the first 100 integers.[1]

Solution We substitute $n = 100$ into $S = \frac{1}{2}n(n+1)$, giving $\frac{1}{2} \cdot 100 \cdot 101 = 50 \cdot 101 = 5050$. ▲

Example 5 Find the sum $4 + 5 + 6 + \cdots + 29$.

Solution This sum is $\sum_{i=4}^{29} i$. We may write it as a difference $\sum_{i=1}^{29} i - \sum_{i=1}^{3} i$ using either "common sense" or summation property 3. Using formula (5) twice gives

$$\sum_{i=4}^{29} i = \tfrac{1}{2} \cdot 29 \cdot 30 - \tfrac{1}{2} \cdot 3 \cdot 4 = 29 \cdot 15 - 3 \cdot 2 = 435 - 6 = 429. \blacktriangle$$

Example 6 Find $\sum_{j=3}^{102}(j-2)$.

Solution We use the summation properties as follows:

$$\begin{aligned}
\sum_{j=3}^{102}(j-2) &= \sum_{j=3}^{102} j - \sum_{j=3}^{102} 2 && \text{(property 1)} \\
&= \sum_{j=1}^{102} j - \sum_{j=1}^{2} j - 2(100) && \text{(properties 3 and 4)} \\
&= \tfrac{1}{2}(102)(103) - 3 - 200 && \text{(formula (5))} \\
&= 5050.
\end{aligned}$$

We can also do this problem by making the substitution $i = j - 2$. As j runs from 3 to 102, i runs from 1 to 100, and we get

$$\sum_{j=3}^{102}(j-2) = \sum_{i=1}^{100} i = \tfrac{1}{2} \cdot 100 \cdot 101 = 5050. \blacktriangle$$

The second method used in Example 6 is usually best carried out by thinking about the meaning of the notation in a given problem. However, for reference, we record the general formula for *substitution of an index*: With the substitution $i = j + q$,

$$\sum_{j=m}^{n} a_{j+q} = \sum_{i=m+q}^{n+q} a_i . \tag{6}$$

The following example illustrates a trick that utilizes cancellation.

Example 7 Show that $\sum_{i=1}^{n}[i^3 - (i-1)^3] = n^3$.

Solution The easiest way to do this is by writing out the sum:

$$\begin{aligned}
\sum_{i=1}^{n}\left[i^3 - (i-1)^3\right] &= \left[1^3 - 0^3\right] + \left[2^3 - 1^3\right] + \left[3^3 - 2^3\right] + \left[4^3 - 3^3\right] \\
&\quad + \cdots + \left[(n-1)^3 - (n-2)^3\right] + \left[n^3 - (n-1)^3\right]
\end{aligned}$$

and observing that we can cancel 1^3 with -1^3, 2^3 with -2^3, 3^3 with -3^3, and so on up to $(n-1)^3$ with $-(n-1)^3$. This leaves only the terms

$$-0^3 + n^3 = n^3. \blacktriangle$$

[1] A famous story about the great mathematician C. F. Gauss (1777–1855) concerns a task his class had received from a demanding teacher in elementary school. They were to add up the first 100 numbers. Gauss wrote the answer 5050 on his slate immediately; had he derived $S = \frac{1}{2}n(n+1)$ in his head at age 10?

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The kind of sum encountered in Example 7 is called a *telescoping*, or *collapsing*, sum. A similar argument proves the result in the following box.

Telescoping Sum

$$\sum_{i=1}^{n} (a_i - a_{i-1}) = a_n - a_0 \tag{7}$$

The next example uses summation notation to retrieve a result which may already be obvious, but the idea will reappear later in the fundamental theorem of calculus.

Example 8 Suppose that the bus in Fig. 4.1.1 is at position y_i at time t_i, $i = 0, \ldots, n$, and that during time interval (t_{i-1}, t_i), the velocity is a constant

$$v_i = \frac{y_i - y_{i-1}}{t_i - t_{i-1}} = \frac{\Delta y_i}{\Delta t_i}, \qquad i = 1, \ldots, n.$$

Using a telescoping sum, confirm that the distance travelled equals the difference between the final and initial position.

Solution By formula (3′), the distance travelled is

$$\Delta d = \sum_{i=1}^{n} v_i \, \Delta t_i .$$

Since $v_i = \Delta y_i / \Delta t_i$, we get

$$\Delta d = \sum_{i=1}^{n} \left(\frac{\Delta y_i}{\Delta t_i} \right) \Delta t_i = \sum_{i=1}^{n} \Delta y_i = \sum_{i=1}^{n} (y_i - y_{i-1}).$$

This is a telescoping sum which, by (7), equals $y_n - y_0$; i.e., the final position minus the initial position (see Fig. 4.1.2 where $n = 3$). ▲

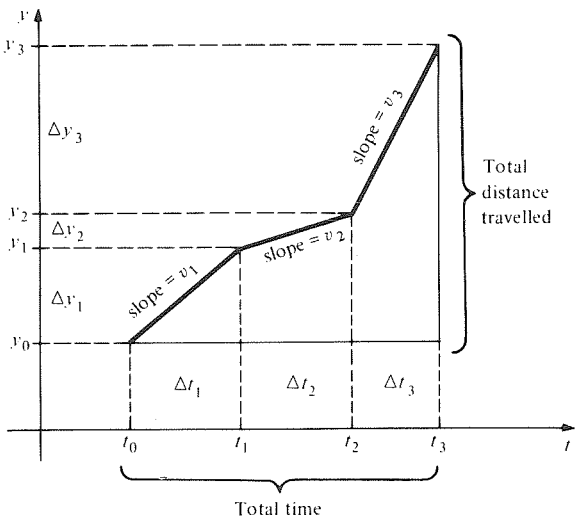

Figure 4.1.2. Motion of the bus in Example 8 ($n = 3$).

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Exercises for Section 4.1

In Exercises 1–4, a particle moves along a line with the given velocities for the given time intervals. Compute the total distance travelled.

1. 2 meters per second for the first 3 seconds,
   1.8 meters per second for the second 2 seconds,
   2.1 meters per second for the third 3 seconds,
   3 meters per second for the fourth 1.5 seconds.
2. 3 meters per second for the first 1.5 seconds,
   1.2 meters per second for the second 3 seconds,
   2.1 meters per second for the third 2.4 seconds,
   4 meters per second for the fourth 3 seconds.
3. 8 meters per second for the first 1.2 seconds,
   10 meters per second for the second 3.1 seconds,
   12 meters per second for the third 4.2 seconds.
4. 2 meters per second for the first 8.1 seconds,
   3.2 meters per second for the second 2 seconds,
   4.6 meters per second for the third 1.1 seconds.

Find the sums in Exercises 5–8.

5. $\sum_{i=1}^{4}(i^2+1)$
6. $\sum_{i=1}^{3} i^3$
7. $\sum_{i=1}^{5} i(i-1)$
8. $\sum_{i=1}^{6} i(i-2)$

Find the sums in Exercises 9–12.

9. $1+2+\cdots+25$
10. $3+4+\cdots+39$
11. $\sum_{i=1}^{45} i$
12. $\sum_{i=3}^{99} i$

Find the sums in Exercises 13–16.

13. $\sum_{j=4}^{80}(j-3)$
14. $\sum_{j=8}^{108}(j-7)$
15. $\sum_{i=1}^{99}[(i+1)^2-i^2]$
16. $\sum_{i=1}^{100}[(i+1)^5-i^5]$
17. Find $\sum_{j=-2}^{2} j^3$.
18. Find $\sum_{j=-1000}^{1000} j^5$.
19. Find $\sum_{j=1}^{102}(j+6)$.
20. Find $\sum_{k=-20}^{10} k$.
21. Find a formula for $\sum_{i=m}^{n} i$, where m and n are positive integers.
22. Find $\sum_{i=1}^{12} a_i$, where a_i is the number of days in the ith month of 1987.
23. Show that $\sum_{k=1}^{1000} 1/(1+k^2) \leqslant 1000$.
24. Show that $\sum_{i=1}^{300} 3/(1+i) \leqslant 300$.

Find the telescoping sums in Exercises 25–28.

25. $\sum_{i=1}^{100}[i^4-(i-1)^4]$
26. $\sum_{i=1}^{32}\{(3i)^2-[3(i-1)]^2\}$
27. $\sum_{i=1}^{100}[(i+2)^2-(i+1)^2]$
28. $\sum_{i=1}^{41}[(i+3)^2-(i+2)^2]$
29. Draw a graph like Fig. 4.1.2 for the data in Exercise 1.
30. Draw a graph like Fig. 4.1.2 for the data in Exercise 2.

Find the sums in Exercises 31–40.

31. $\sum_{k=0}^{100}(3k-2)$
32. $\sum_{i=0}^{n}(2i+1)$
33. $\sum_{k=1}^{n}[(k+1)^4-k^4]$
34. $\sum_{k=1}^{300}[(k+1)^8-k^8]$
35. $\sum_{i=1}^{100}[(i+2)^2-(i-1)^2]$
36. $\sum_{i=1}^{50}[(2i+2)^3-(2i)^3]$
37. $\sum_{i=-30}^{30}[i^5+i+2]$
38. $\sum_{l=-75}^{75}[l^9+5l^7-13l^5+l]$
39. $\sum_{j=2}^{6} 2^j$
40. $\sum_{k=1}^{3} 3^k$

★41. By the method of telescoping sums, we have

$$\sum_{i=1}^{n}\left[(i+1)^3-i^3\right]=(n+1)^3-1.$$

(a) Write $(i+1)^3-i^3=3i^2+3i+1$ and use properties of summation to prove that

$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}.$$

(b) Find a formula for

$$\sum_{i=m}^{n} i^2$$

in terms of m and n.

(c) Using the *method* and result of (a), find a formula for $\sum_{i=1}^{n} i^3$. (You may wish to try guessing an answer by experiment.)

★42. (a) Prove that

$$\sum_{i=1}^{n} i(i+1)=\frac{1}{3}n(n+1)(n+2)$$

by writing

$$i(i+1)=\frac{1}{3}[i(i+1)(i+2)-(i-1)i(i+1)]$$

and using a telescoping sum.

(b) Find $\sum_{i=1}^{n} i(i+1)(i+2)$.

(c) Find $\sum_{i=1}^{n}[1/i(i+1)]$.

4.2 Sums and Areas

Areas under graphs can be approximated by sums.

In the last section, we saw that the formula for distance in terms of velocity is $\Delta d=\sum_{i=1}^{n} v_i\Delta t_i$ when the velocity is a constant v_i during the time interval (t_{i-1}, t_i). In this section we shall discuss a geometric interpretation of this fact which will be important in the study of integration.

Let us plot the velocity of a bus as a function of time. Suppose that the

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total time interval in question is $[a,b]$; i.e., t runs from a to b, and this interval is divided into n smaller intervals so that $a = t_0 < t_1 < \cdots < t_n = b$. The ith interval is (t_{i-1}, t_i), and v is a constant v_i on this interval.[2] The form of v is shown in Fig. 4.2.1 for $n = 5$.

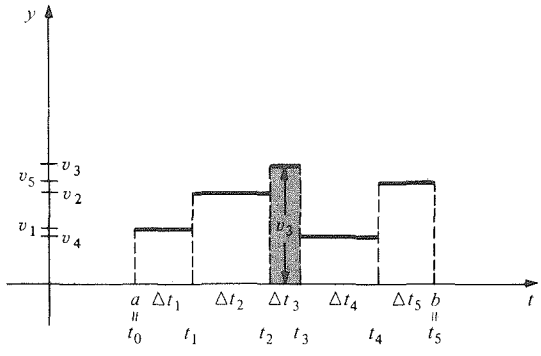

Figure 4.2.1. The velocity of the bus.

We notice that $v_i \Delta t_i$ is exactly the area of the rectangle over the ith interval with base Δt_i and height v_i (the rectangle for $i = 3$ is shaded in the figure). Thus,

$$\Delta d = \sum_{i=1}^{n} v_i \Delta t_i \text{ is the total area of the rectangles under the graph of } v.$$

This suggests that the problem of finding distances in terms of velocities should have something to do with areas, even when the velocity changes smoothly rather than abruptly. Turning our attention to areas then, we go back to the usual symbol x (rather than t) for the independent variable.

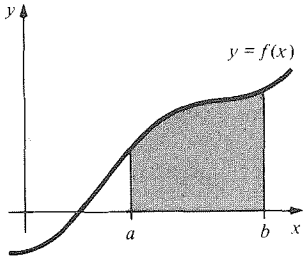

Figure 4.2.2. The region under the graph of f on $[a,b]$.

The *area under the graph* of a function f on an interval $[a,b]$ is defined to be the area of the region in the plane enclosed by the graph $y = f(x)$, the x axis, and the vertical lines $x = a$ and $x = b$. (See Fig. 4.2.2.) Here we assume that $f(x) \geqslant 0$ for x in $[a,b]$. (In the next section, we shall deal with the possibility that f might take negative values.)

Let us examine certain similarities between properties of sums and areas. To the property $\sum_{i=m}^{p} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i$ of sums, there corresponds the additive property of areas: if a plane region is split into two parts which overlap only along their edges, the area of the region is the sum of the areas of the parts. (See Fig. 4.2.3.) Another property of sums is that if $a_i \leqslant b_i$ for

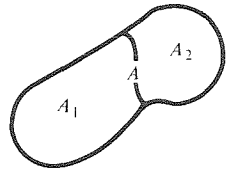

Figure 4.2.3. Area (A) = Area (A_1) + Area (A_2).

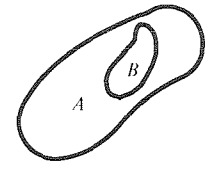

Figure 4.2.4. Area $(A) \geqslant$ Area (B).

[2] We are deliberately vague about the value of v at the end points, when the bus must suddenly switch velocities. The value of Δd does not depend on what v is at each t_i, so we can safely ignore these points.

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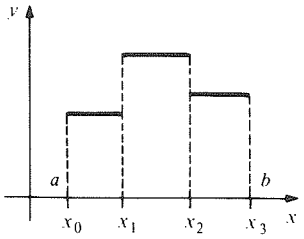

Figure 4.2.5. The graph of a step function on $[a, b]$ with $n = 3$.

$i = m, m+1, \ldots, n$, then $\sum_{i=m}^{n} a_i \leqslant \sum_{i=m}^{n} b_i$; the counterpart for areas is the inclusion property: if one plane region is contained in another, the containing region has more area. (See Fig. 4.2.4.)

The connection between areas and sums becomes more explicit if we consider step functions. A function g on the interval $[a, b]$ is called a *step function* if $[a, b]$ can be broken into smaller intervals (called subintervals) with g constant on each part. More precisely, there should be numbers $x_0, x_1, \ldots, x_n$, with $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, such that g is constant on each of the intervals $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$, as in Fig. 4.2.5. The values of g at the endpoints of these intervals will not affect any of our calculations. The list $(x_0, x_1, \ldots, x_n)$ is called a *partition* of $[a, b]$.

Example 1 Draw a graph of the step function g defined on $[2, 4]$ by

$$g(x) = \begin{cases} 1 & \text{if } 2 \leqslant x \leqslant 2.5, \\ 3 & \text{if } 2.5 < x < 3.5, \\ 2 & \text{if } 3.5 \leqslant x \leqslant 4. \end{cases}$$

Solution The graph of g on $[2, 2.5]$ is a horizontal line with height 1 on this interval. The endpoints on the graph are drawn as solid dots to indicate that g takes the value 1 at the endpoints $x = 2$ and $x = 2.5$. Continuing through the remaining subintervals and using open dots to indicate endpoints which do not belong to the graph, we obtain Fig. 4.2.6. ▲

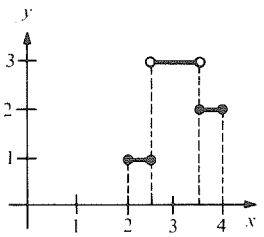

Figure 4.2.6. The graph of the step function g in Example 1.

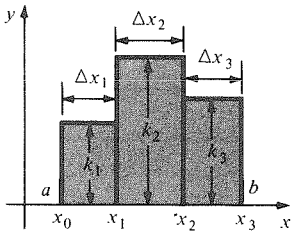

Figure 4.2.7. The shaded area is the sum of $k_1 \Delta x_1$, $k_2 \Delta x_2$ and $k_3 \Delta x_3$.

If a step function is non-negative, then the region under its graph can be broken into rectangles, and the area of the region can be expressed as a sum. It is common to write Δx_i for length $x_i - x_{i-1}$ of the ith partition interval; if the value of g on this interval is $k_i \geqslant 0$, then the area of the rectangle from x_{i-1} to x_i with height k_i is $k_i \Delta x_i$. Thus the total area under the graph is $k_1 \Delta x_1 + k_2 \Delta x_2 + \cdots + k_n \Delta x_n = \sum_{i=1}^{n} k_i \Delta x_i$, as in Fig. 4.2.7.

Example 2 What are the x_i's, Δx_i's, and k_i's for the step function in Example 1? Compute the area of the region under its graph.

Solution Looking at Figs. 4.2.6 and 4.2.7, we begin by labelling the left endpoint as x_0; i.e., $x_0 = 2$. The remaining partition points are $x_1 = 2.5$, $x_2 = 3.5$, and $x_3 = 4$. The Δx_i's are the widths of the intervals: $\Delta x_1 = x_1 - x_0 = 0.5$, $\Delta x_2 = 1$, and $\Delta x_3 = 0.5$. Finally, the k_i's are the heights of the rectangles: $k_1 = 1$, $k_2 = 3$, and $k_3 = 2$. The area under the graph is

$$\sum_{i=1}^{3} k_i \Delta x_i = (1)(0.5) + (3)(1) + (2)(0.5) = 4.5. \quad \blacktriangle$$

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Step Functions

A function g on $[a,b]$ is a step function when $[a,b]$ can be broken up into intervals of width Δx_i, on each of which g equals a constant k_i.

If each k_i is positive (or zero), the area under the graph of g is $\sum_{i=1}^{n} k_i \Delta x_i$.

In deriving our formula for the area under the graph of a step function, we used the fact that the area of a rectangle is its length times width, and the additive property of areas. By using the inclusion property, we can find the areas under graphs of *general* functions by comparison with step functions—this idea, which goes back to the ancient Greeks, is the key to defining the integral.

Given a non-negative function f, we wish to compute the area A under its graph on $[a,b]$. A *lower sum* for f on $[a,b]$ is defined to be the area under the graph of a non-negative step function g for which $g(x) \leqslant f(x)$ on $[a,b]$. If $g(x) = k_i$ on the ith subinterval, then the inclusion property of areas tells us that $\sum_{i=1}^{n} k_i \Delta x_i \leqslant A$. (Fig. 4.2.8).

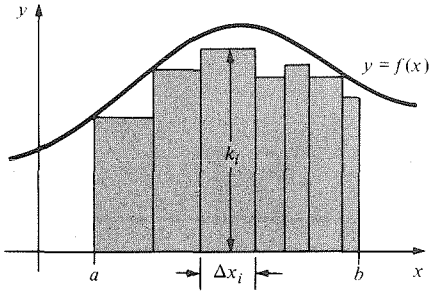

Figure 4.2.8. The shaded area $\sum_{i=1}^{n} k_i \Delta x_i$ is a lower sum for f on $[a,b]$.

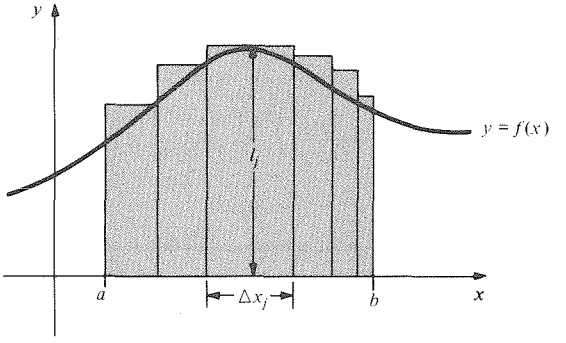

Figure 4.2.9. The shaded area $\sum_{j=1}^{m} l_j \Delta x_j$ is an upper sum for f on $[a,b]$.

Similarly, an *upper sum* for f on $[a,b]$ is defined to be $\sum_{j=1}^{m} l_j \Delta x_j$, where h is a step function with $f(x) \leqslant h(x)$ on $[a,b]$, and $h(x) = l_j$ on the jth subinterval of a partition of $[a,b]$ (Fig. 4.2.9). By the inclusion property for areas, $A \leqslant \sum_{j=1}^{m} l_j \Delta x_j$, so the area lies between the upper and lower sums.

Example 3 Let $f(x) = x^2 + 1$ for $0 \leqslant x \leqslant 2$. Let

$$g(x) = \begin{cases} 0 & 0 \leqslant x \leqslant 1 \\ 2 & 1 < x \leqslant 2 \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 2 & 0 \leqslant x \leqslant \frac{2}{3}, \\ 4 & \frac{2}{3} < x \leqslant \frac{4}{3}, \\ 5 & \frac{4}{3} < x \leqslant 2. \end{cases}$$

Draw a graph showing $f(x)$, $g(x)$, and $h(x)$. What upper and lower sums for f can be obtained from g and h?

Solution The graphs are shown in Fig. 4.2.10.

For g we have $\Delta x_1 = 1$, $k_1 = 0$ and $\Delta x_2 = 1$, $k_2 = 2$. Since $g(x) \leqslant f(x)$ for all x in the open interval $(0,2)$ (the graph of g lies below that of f), we have as a lower sum,

$$\sum_{i=1}^{2} k_i \Delta x_i = 0 \cdot 1 + 2 \cdot 1 = 2.$$

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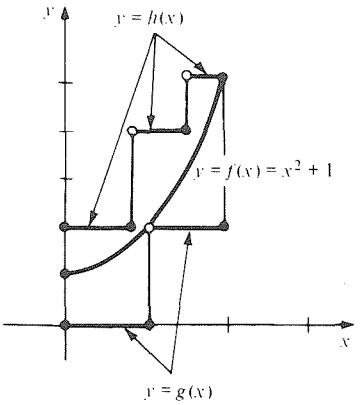

Figure 4.2.10. The area under the graph of h is an upper sum for f; the area under the graph of g is a lower sum.

For h we have $\Delta x_1 = \frac{2}{3}$, $l_1 = 2$, $\Delta x_2 = \frac{2}{3}$, $l_2 = 4$, and $\Delta x_3 = \frac{2}{3}$, $l_3 = 5$. Since the graph of h lies above that of f, $h(x) \geqslant f(x)$ for all x in the interval $(0, 2)$, we get the upper sum

$$\sum_{i=1}^{3} l_i \Delta x_i = 2 \cdot \tfrac{2}{3} + 4 \cdot \tfrac{2}{3} + 5 \cdot \tfrac{2}{3} = \tfrac{22}{3} = 7\tfrac{1}{3}. \blacktriangle$$

Using partitions with sufficiently small subintervals, we hope to find step functions below and above f such that the corresponding lower and upper sums are as close together as we wish. Notice that the difference between lower and upper sums is the area *between* the graphs of the step functions (Fig. 4.2.11). We expect this area to be very small if the subintervals are small enough and the values of the step functions are close to the values of f.

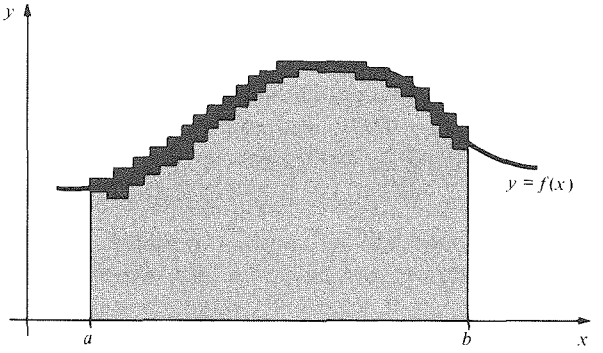

Figure 4.2.11. The dark shaded area is the difference between upper and lower sums for f on $[a, b]$. The area under the graph is between the upper and lower sums.

Suppose that there are lower sums and upper sums which are arbitrarily close to one another. Then there can only be one number A such that $L \leqslant A \leqslant U$ for every lower sum L and every upper sum U, and this number must be the area under the graph.

Area Under a Graph

To calculate the area under the graph of a non-negative function f, we try to find upper and lower sums (areas under graphs of step functions lying below and above f) which are closer and closer together. (See Example 6 below for a specific instance of what is meant by "closer and closer.") The area A is the number which is above all the lower sums and below all the upper sums.

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What we have done here for areas has a counterpart in our distance–velocity problem. Suppose that $v = f(t)$ defined for $a \leqslant t \leqslant b$ gives the velocity of a moving bus as a function of time, and that there is a partition $(t_0, t_1, \ldots, t_n)$ of $[a, b]$ and numbers $k_1, \ldots, k_n$ such that $k_i \leqslant f(t)$ for t in the ith interval (t_{i-1}, t_i). Taking for granted that a faster moving object travels further in a given time interval, we may conclude that the bus travels a distance at least $k_i(t_i - t_{i-1})$ in the ith time interval. Thus the total distance travelled must be *at least* $k_1 \Delta t_1 + \cdots + k_n \Delta t_n = \sum_{i=1}^{n} k_i \Delta t_i$ (where, as usual, we write Δt_i for $t_i - t_{i-1}$), so we have a *lower estimate* for the distance travelled between $t = a$ and $t = b$. Similarly, if we know that $f(t) \leqslant l_i$ on (t_{i-1}, t_i) for some numbers $l_1, \ldots, l_n$, we get an *upper estimate* $\sum_{i=1}^{n} l_i \Delta t_i$ for the distance travelled. By making the time intervals short enough we hope to be able to find k_i and l_i close together, so that we can estimate the distance travelled as accurately as we wish.

Example 4 The velocity of a moving bus (in meters per second) is observed over periods of 10 seconds, and it is found that

$$\begin{aligned} 4 \leqslant v \leqslant 5 \quad &\text{when} \quad 0 < t < 10, \\ 5.5 \leqslant v \leqslant 6.5 \quad &\text{when} \quad 10 < t < 20, \\ 5 \leqslant v \leqslant 5.7 \quad &\text{when} \quad 20 < t < 30. \end{aligned}$$

Estimate the distance travelled during the interval $0 \leqslant t \leqslant 30$.

Solution A lower estimate is $4 \cdot 10 + 5.5 \cdot 10 + 5 \cdot 10 = 145$, and an upper estimate is $5 \cdot 10 + 6.5 \cdot 10 + 5.7 \cdot 10 = 172$, so the distance travelled is between 145 and 172 meters. ▲

Example 5 The velocity of a snail at time t seconds is $(0.001)(t^2 + 1)$ meters per second at time t. Use the calculations in Example 3 to estimate how far the snail crawled between $t = 0$ and $t = 2$.

Solution We may use the comparison functions g and h in Example 3 if we multiply their values by 0.001 (and change x to t). The lower sum and upper sum are also multiplied by 0.001, and so the distance crawled is between 0.002 and 0.00733 . . . meters, i.e., between 2 and $7\frac{1}{3}$ millimeters. ▲

When we calculate derivatives, we seldom use the definition in terms of limits. Rather, we use the rules for derivatives, which are much more efficient. Likewise, we will not usually calculate areas in terms of upper and lower sums but will use the fundamental theorem of calculus once we have learned it. Now, however, to reinforce the idea of upper and lower sums, we shall do one area problem "the hard way."

Example 6 Use upper and lower sums to find the area under the graph of $f(x) = x$ on $[0, 1]$.

Solution The area is shaded in Fig. 4.2.12.

We will look for upper and lower sums which are close together. The simplest way to do this is to divide the interval $[0, 1]$ into equal parts with a partition of the form $(0, 1/n, 2/n, \ldots, (n-1)/n, 1)$. A step function $g(x)$ below $f(x)$ is given by setting $g(x) = (i-1)/n$ on the interval $[(i-1)/n, i/n)$, while the step function with $h(x) = i/n$ on $((i-1)/n, i/n]$ is above $f(x)$ (Fig. 4.2.13).

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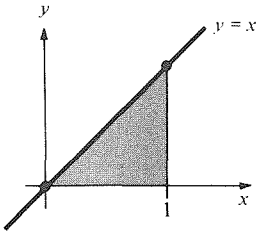


Figure 4.2.12. The region under the graph of $f(x) = x$ on $[0, 1]$.

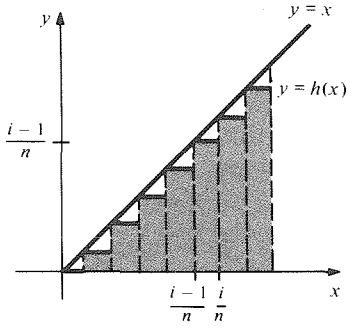
The difference between the upper and lower sums is equal to the total area of the chain of boxes in Fig. 4.2.14, on which both $g(x)$ and $h(x)$ are graphed. Each of the n boxes has area $(1/n) \cdot (1/n) = 1/n^2$, so their total area is $n \cdot (1/n^2) = 1/n$, which becomes arbitrarily small as $n \to \infty$, so we know that the area under our graph will be precisely determined. To find the area, we compute the upper and lower sums. For the lower sum, $g(x) = (i-1)/n = k_i$ on the ith subinterval, and $\Delta x_i = 1/n$ for all i, so

$$\sum_{i=1}^{n} k_i \Delta x_i = \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \frac{1}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right)$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right] = \frac{n+1}{2n} - \frac{1}{n} = \frac{1}{2} - \frac{1}{2n}.$$

The upper sum is

$$\sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}.$$



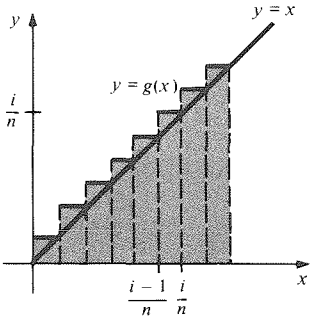


Figure 4.2.13. Lower and upper sums for $f(x) = x$ on $[0, 1]$.

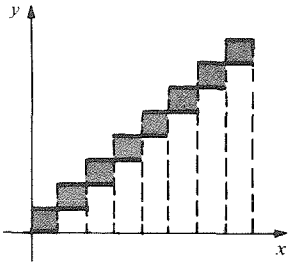


Figure 4.2.14. Difference between the upper and lower sums.

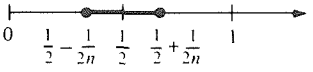


Figure 4.2.15. The area lies in the interval $[1/2 - 1/2n, 1/2 + 1/2n]$ for all n. The length of this interval $\to 0$ as $n \to \infty$.

The area under the graph is the unique number A which satisfies the inequalities $1/2 - 1/2n \leq A \leq 1/2 + 1/2n$ *for all* n (see Fig. 4.2.15). Since the number $\frac{1}{2}$ satisfies the condition, we must have $A = \frac{1}{2}$. ▲

The result of Example 6 agrees with the rule from elementary geometry that the area of a triangle is half the base times the height. The advantage of the method used here is that it can be applied to more general graphs. (Another case is given in Exercise 20.) This method was used extensively during the century before the invention of calculus, and is the basis for the definition of the integral.

Exercises for Section 4.2

Draw the graphs of the step functions in Exercises 1–4.

1. $g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ 2 & \text{if } 1 < x < 2, \\ 1 & \text{if } 2 \leq x \leq 3. \end{cases}$

2. $g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 0.5, \\ 3 & \text{if } 0.5 < x < 2, \\ 2 & \text{if } 2 \leq x \leq 3, \\ 4 & \text{if } 3 < x \leq 4. \end{cases}$

3. $g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } 1 \leq x < 2, \\ 2 & \text{if } 2 \leq x \leq 3. \end{cases}$

4. $g(x) = \begin{cases} 1 & \text{if } 2 \leq x < 2.5, \\ 3 & \text{if } 2.5 \leq x \leq 3, \\ 4 & \text{if } 3 < x < 4, \\ 0 & \text{if } 4 \leq x \leq 4.5. \end{cases}$

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In Exercises 5–8 compute the x_i's, Δx_i's, and k_i's for the indicated step function and compute the area of the region under its graph.

5. For g in Exercise 1.
6. For g in Exercise 2.
7. For g in Exercise 3.
8. For g in Exercise 4.

In Exercises 9 and 10, draw a graph showing f, g, and h and compute the upper and lower sums for f obtained from g and h.

9. $f(x) = x^2$, $1 \leqslant x \leqslant 3$;

$$g(x) = \begin{cases} 1, & 1 \leqslant x \leqslant 2, \\ 4, & 2 < x \leqslant 3; \end{cases}$$

$$h(x) = \begin{cases} 4, & 1 \leqslant x < 2, \\ 9, & 2 \leqslant x \leqslant 3. \end{cases}$$

10. $f(x) = x^3 + 1$, $1 \leqslant x \leqslant 3$;

$$g(x) = \begin{cases} 2, & 1 \leqslant x \leqslant 1.5, \\ 4, & 1.5 < x < 2, \\ 9, & 2 \leqslant x \leqslant 3; \end{cases}$$

$$h(x) = \begin{cases} 9, & 1 \leqslant x \leqslant 2, \\ 28, & 2 < x \leqslant 3. \end{cases}$$

11. The velocity of a moving bus (in meters per second) is observed over periods of 5 seconds and it is found that

$$\begin{array}{lll} 5.0 \leqslant v \leqslant 6.0 & \text{when} & 0 < t < 5, \\ 4.0 \leqslant v \leqslant 5.5 & \text{when} & 5 < t < 10, \\ 6.1 \leqslant v \leqslant 7.2 & \text{when} & 10 < t < 15, \\ 3.2 \leqslant v \leqslant 4.7 & \text{when} & 15 < t < 20. \end{array}$$

Estimate the distance travelled during the interval $t = 0$ to $t = 20$.

12. The velocity of a moving bus (in meters per second) is observed over periods of 7.5 seconds and it is found that

$$\begin{array}{lll} 4.0 \leqslant v \leqslant 5.1 & \text{when} & 0 < t < 7.5, \\ 3.0 \leqslant v \leqslant 5.0 & \text{when} & 7.5 < t < 15, \\ 4.4 \leqslant v \leqslant 5.5 & \text{when} & 15 < t < 22.5, \\ 3.0 \leqslant v \leqslant 4.1 & \text{when} & 22.5 < t < 30. \end{array}$$

Estimate the distance travelled during the interval $t = 0$ to $t = 30$.

13. The velocity of a snail at time t is $(0.002)t^2$ meters per second at time t. Use the functions g and h in Exercise 9 to estimate how far the snail crawled between $t = 1$ and $t = 3$.
14. The velocity of a snail at time t is given by $(0.0005)(t^3 + 1)$ meters per second at time t. Use the functions g and h in Exercise 10 to estimate how far the snail crawled between $t = 1$ and $t = 3$.

In Exercises 15–18, use upper and lower sums to find the area under the graph of the given function.

15. $f(x) = x$ for $1 \leqslant x \leqslant 2$.
16. $f(x) = 2x$ for $0 \leqslant x \leqslant 1$.
17. $f(x) = 5x$ for $a \leqslant x \leqslant b$, $a > 0$.
18. $f(x) = x + 3$ for $a \leqslant x \leqslant b$, $a > 0$.
19. Using upper and lower sums, find the area under the graph of $f(x) = 1 - x$ between $x = 0$ and $x = 1$.

★20. Using upper and lower sums, show that the area under the graph of $f(x) = x^2$ between $x = a$ and $x = b$ is $\frac{1}{3}(b^3 - a^3)$. (You will need to use the result of Exercise 41(a) from Section 4.1.)

★21. Let $f(x) = \begin{cases} x^2, & 0 \leqslant x < 1, \\ x, & 1 \leqslant x \leqslant 2. \end{cases}$ Find the area under the graph of f on $[0, 2]$, using the results of the Exercises 15 and 20.

★22. Let

$$f(x) = \begin{cases} 1 - x, & 0 \leqslant x < 1, \\ 5x, & 1 \leqslant x \leqslant 4. \end{cases}$$

Using the results of Exercises 17 and 19, find the area under the graph of f on $[0, 4]$.

★23. By combining the results of Example 6 and Exercise 20, find the area of the shaded region in Fig. 4.2.16. (*Hint:* Write the area as a difference of known areas.)

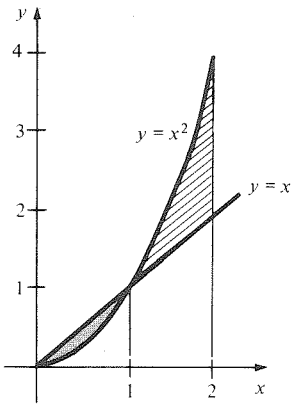

Figure 4.2.16. Find the shaded and striped areas.

★24. Using the results of previous exercises, find the area of the striped region in Fig. 4.2.16.

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4.3 The Definition of the Integral

The integral of a function is a "signed" area.

In the previous section, we saw how areas under graphs could be approximated by the areas under graphs of step functions. Now we shall extend this idea to functions that need not be positive and shall give the formal definition of the integral.

Recall that if g is a step function with constant value $k_i \geqslant 0$ on the interval (x_{i-1}, x_i) of width $\Delta x_i = x_i - x_{i-1}$, then the area under the graph of g is

$$\text{Area} = \sum_{i=1}^{n} k_i \Delta x_i .$$

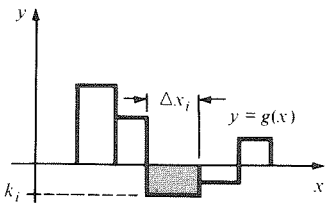

Figure 4.3.1. The product $k_i \Delta x_i$ is the negative of the shaded area.

This formula is analogous to the formula for distance travelled when the velocity is a step function; see formula (3), Section 4.1. In that situation, it is reasonable to allow negative velocity (reverse motion). Likewise, in the area formula we wish to allow negative k_i. To do so, we shall have to interpret "area" correctly. Suppose that $g(x)$ is a *negative* constant k_i on an interval of width Δx_i. Then $k_i \Delta x_i$ is the *negative* of the area between the graph of g and the x axis on that interval. (See Fig. 4.3.1.)

To formalize this idea, we introduce the notion of signed area. If R is any region in the xy plane, its *signed area* is defined to be the area of the part of R lying above the x axis, minus the area of the part lying below the axis.

If f is a function defined on the interval $[a, b]$, the region between the graph of f and the x axis consists of those points (x, y) for which x is in $[a, b]$ and y lies betwen 0 and $f(x)$. It is natural to consider the signed area of such a region, as illustrated in Fig. 4.3.2. For a step function g with values k_i on intervals of length Δx_i, the sum $\sum_{i=1}^{n} k_i \Delta x_i$ gives the signed area of the region between the graph of g and the x axis.

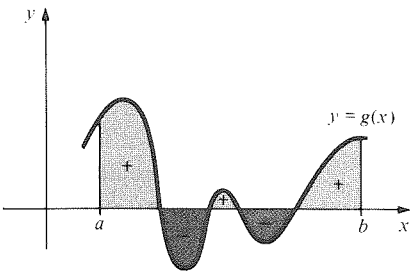

Figure 4.3.2. The signed area between the graph of f and the x axis on $[a, b]$ is the area of the + regions minus the area of the − regions.

Signed Area

The signed area of a region is the area of the portion above the x axis minus the area of the portion below the x axis.

For the region between the x axis and the graph of a step function g, this signed area is $\sum_{i=1}^{n} k_i \Delta x_i$.

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Example 1 Draw a graph of the step function g on $[0, 1]$ defined by

$$g(x) = \begin{cases} -2 & \text{if } 0 \leqslant x < \frac{1}{3}, \\ 3 & \text{if } \frac{1}{3} \leqslant x \leqslant \frac{3}{4}, \\ 1 & \text{if } \frac{3}{4} < x \leqslant 1. \end{cases}$$

Compute the signed area of the region between its graph and the x axis.

Solution The graph is shown in Fig. 4.3.3. There are three intervals, with $\Delta x_1 = \frac{1}{3}$, $\Delta x_2 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$, and $\Delta x_3 = 1 - \frac{3}{4} = \frac{1}{4}$; $k_1 = -2$, $k_2 = 3$, and $k_3 = 1$. Thus the signed area is

$$\sum_{i=1}^{3} k_i \Delta x_i = (-2)(\tfrac{1}{3}) + (3)(\tfrac{5}{12}) + (1)(\tfrac{1}{4}) = -\tfrac{2}{3} + \tfrac{5}{4} + \tfrac{1}{4} = \tfrac{5}{6}. \blacktriangle$$

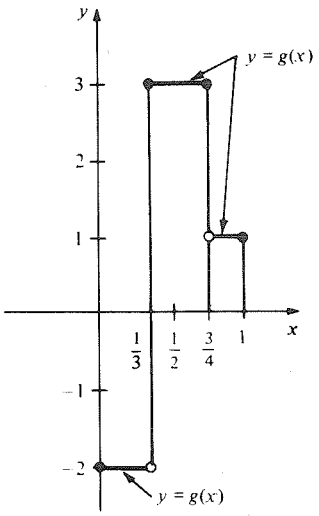

Figure 4.3.3. The graph of the step function in Example 1.

The counterpart of signed area for our distance–velocity problem is directed distance, explained as follows: If the bus is moving to the right, then $v > 0$ and distances are increasing. If the bus is moving to the left, then $v < 0$ and the distances are decreasing. In the formula $\Delta d = \sum_{i=1}^{n} v_i \Delta t_i$, Δd is the displacement, or the net distance the bus has moved, not the total distance travelled, which would be $\sum_{i=1}^{n} |v_i| \Delta t$. Just as with signed areas, movement to the left is considered negative and is subtracted from movement to the right. (See Fig. 4.3.4.)

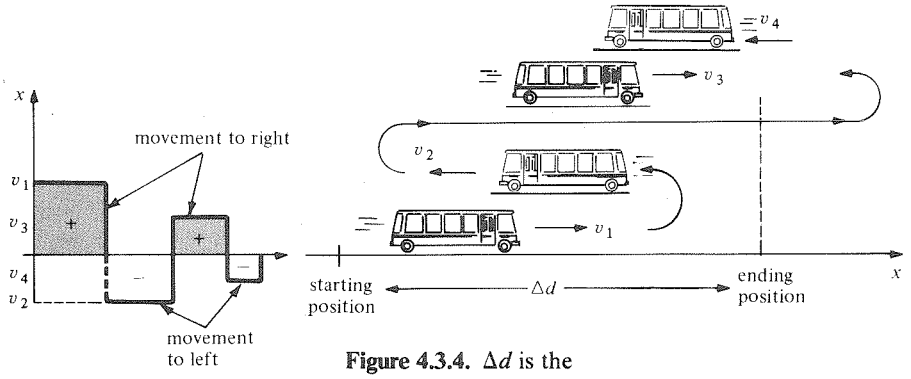

Figure 4.3.4. Δd is the displacement; i.e., net distance travelled.

To find the signed area between the graph and the x axis for a function which is not a step function, we can use upper and lower sums. Just as with positive functions, if g is a step function lying below f, i.e., $g(x) \leqslant f(x)$ for x in $[a, b]$, we call

$$L = \sum_{i=1}^{n} k_i \Delta x_i$$

a *lower sum* for f. Likewise, if h is a step function lying above f, with values l_j on intervals of width Δx_j for $j = 1, \ldots, m$, then

$$U = \sum_{j=1}^{m} l_j \Delta x_j$$

is an *upper sum* for f. If we can find L's and U's arbitrarily close together, lying on either side of a number A, then A must be the *signed area* between the graph of f and the x axis on $[a, b]$. (See Fig. 4.3.5.)

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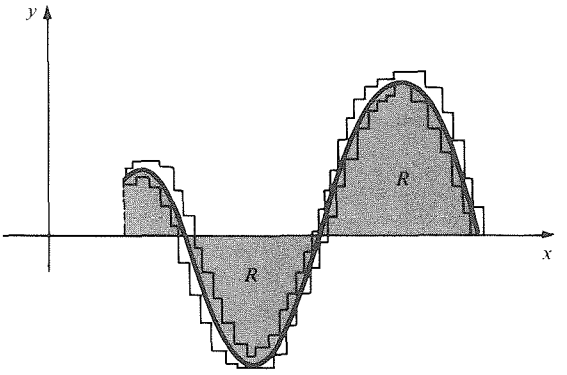

Figure 4.3.5. The signed area of the region R lies between the upper and lower sums.

We are now ready to define the integral of a function f.

Definition Let f be a function defined on $[a,b]$. We say that f *has an integral* or that f *is integrable* if upper and lower sums for f can be found which are arbitrarily close together. The number I such that $L \leqslant I \leqslant U$ for all lower sums L and upper sums U is called the *integral* of f and is denoted

$$\int_a^b f(x)\,dx.$$

We call $\int$ the *integral sign*, a, b, the *endpoints* or *limits of integration*, and f the *integrand*.

The precise meaning of "arbitrarily close together" is the same as in Example 6, Section 4.2, namely, that there should exist sequences L_n and U_n of lower and upper sums such that $\lim_{n\to\infty}(U_n - L_n) = 0$. (Limits of sequences will be treated in detail in Chapter 11.)

The Integral

Given a function f on $[a,b]$, the integral of f, if it exists, is the number

$$\int_a^b f(x)\,dx$$

which separates the upper and lower sums. This number is the signed area of the region between the graph of f and the x axis.

The notation for the integral is derived from the notation for sums. The Greek letter $\sum$ has turned into an elongated S; k_i and l_j have turned into function values $f(x)$; Δx_i has become dx; and the limits of summation (e.g., i goes from 1 to n) have become limits of integration:

$$\sum_{i=1}^{n} k_i \Delta x_i$$

$$\Downarrow$$

$$\int_a^b f(x)\,dx$$

Just as with antiderivatives, the "x" in "dx" indicates that x is the variable of integration.

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Example 2 Compute $\int_{-2}^{3} f(x)\,dx$ for the function f sketched in Fig. 4.3.6. How is the integral related to the area of the shaded region in the figure?

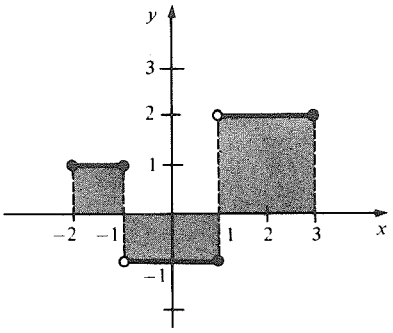

Figure 4.3.6. The signed area of the shaded region is an integral.

Solution The integral $\int_{-2}^{3} f(x)\,dx$ is the signed area of the shaded region.

$$\int_{-2}^{3} f(x)\,dx = (1)(1) + (-1)(2) + (2)(2) = 1 - 2 + 4 = 3. \blacktriangle$$

Example 3 Write the signed area of the region in Fig. 4.3.7 as an integral.

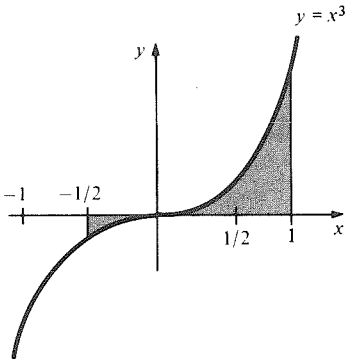

Figure 4.3.7. The signed area of this region equals what integral?

Solution The region is that between the graph of $y = x^3$ and the x axis from $x = -\frac{1}{2}$ to $x = 1$, so the signed area is

$$\int_{-1/2}^{1} x^3\,dx. \blacktriangle$$

The next example shows how upper and lower sums can be used to approximate an integral. (In Chapter 6, we will learn how to compute this integral exactly.)

Example 4 Using a division of the interval $[1,2]$ into three equal parts, find $\int_1^2 (1/x)\,dx$ to within an error of no more than $\frac{1}{10}$.

Solution To estimate the integral within $\frac{1}{10}$, we must find lower and upper sums which are within $\frac{2}{10}$ of one another. We divide the interval into three equal parts and use the step functions which give us the lowest possible upper sum and highest possible lower sum, as shown in Fig. 4.3.8. For a lower sum we have

$$\int_1^2 g(x)\,dx = \frac{1}{4/3}\left(\frac{4}{3} - 1\right) + \frac{1}{5/3}\left(\frac{5}{3} - \frac{4}{3}\right) + \frac{1}{2}\left(2 - \frac{5}{3}\right)$$

$$= \frac{3}{4}\cdot\frac{1}{3} + \frac{3}{5}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{3}{4} + \frac{3}{5} + \frac{1}{2}\right) = \frac{1}{3}\left(\frac{37}{20}\right) = \frac{37}{60}.$$

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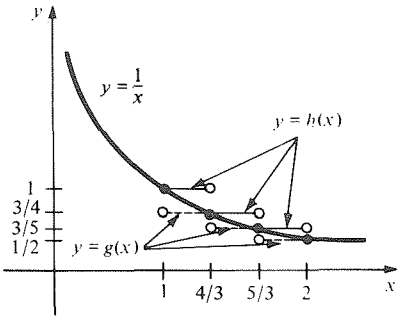

Figure 4.3.8. Illustrating upper and lower sums for $1/x$ on $[1, 2]$.

For an upper sum we have

$$\int_1^2 h(x)\,dx = \frac{1}{1}\cdot\frac{1}{3} + \frac{1}{4/3}\cdot\frac{1}{3} + \frac{1}{5/3}\cdot\frac{1}{3}$$

$$= \frac{1}{3}\left(1 + \frac{3}{4} + \frac{3}{5}\right)$$

$$= \frac{1}{3}\left(\frac{47}{20}\right) = \frac{47}{60}.$$

It follows that

$$\frac{37}{60} \leqslant \int_1^2 \frac{1}{x}\,dx \leqslant \frac{47}{60}.$$

Since the integral lies in the interval $[\frac{37}{60}, \frac{47}{60}]$, whose length is $\frac{1}{6}$, we may take the midpoint $\frac{42}{60} = \frac{7}{10}$ as our estimate; it will differ from the true integral by no more than $\frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}$, which is less than $\frac{1}{10}$. ▲

We have been calculating approximations to integrals without knowing whether some of those integrals actually exist or not. Thus it may be reassuring to know the following fact whose proof is given in more advanced courses.[3]

Existence Theorem *If f is continuous on $[a, b]$, then it has an integral.*

In particular, all differentiable functions have integrals, but so do step functions and functions whose graphs have corners (such as $y = |x|$); thus, integrability is a more easily satisfied requirement than differentiability or even continuity. The reader should note, however, that there do exist some "pathological" functions that are not integrable. (See Exercise 36).

It is possible to calculate integrals of functions which are not necessarily positive by the method used in Example 6 of the previous section, but this is a tedious process. Rather than doing any such examples here, we shall wait until we have developed the machinery of the fundamental theorem of calculus to assist us.

Let us now interpret the integral in terms of the distance–velocity problem. We saw in our previous work that the upper and lower sums represent the displacement of vehicles whose velocities are step functions and which are faster or slower than the one we are studying. Thus, the displace-

[3] See, for instance, *Calculus Unlimited* by J. Marsden and A. Weinstein, Benjamin/Cummings (1981), p. 159, or one of the other references given in the Preface.

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ment, like the integral, is sandwiched between upper and lower sums for the velocity function, so we must have

$$\text{displacement} = \int_a^b f(t)\,dt.$$

Example 5 A bus moves on the line with velocity $v = (t^2 - 4t + 3)$ meters per second. Write formulas in terms of integrals for:

(a) the displacement of the bus between $t = 0$ and $t = 3$;
(b) the actual distance the bus travels between $t = 0$ and $t = 3$.

Solution (a) The displacement is $\int_0^3 (t^2 - 4t + 3)\,dt$.
(b) We note that v can be factored as $(t - 1)(t - 3)$, so it is positive on $(0, 1)$ and negative on $(1, 3)$. The total distance travelled is thus

$$\int_0^1 (t^2 - 4t + 3)\,dt - \int_1^3 (t^2 - 4t + 3)\,dt. \blacktriangle$$

We close this section with a discussion of a different approach to the integral, called the *method of Riemann sums*. Later we shall usually rely on the step function approach, but Riemann sums are also widely used, and so you should have at least a brief exposure to them.

The idea behind Riemann sums is to use step functions to *approximate* the function to be integrated, rather than bounding it above and below. Given a function f defined on $[a, b]$ and a partition $(x_0, x_1, \ldots, x_n)$ of that interval, we choose points $c_1, \ldots, c_n$ such that c_i lies in the interval $[x_{i-1}, x_i]$. The step function which takes the constant value $f(c_i)$ on (x_{i-1}, x_i) is then an approximation to f; the signed area under its graph, namely,

$$S_n = \sum_{i=1}^{n} f(c_i)\,\Delta x_i,$$

is called a Riemann sum.[4] It lies above all the lower sums and below all the upper sums constructed using the same partition, so it is a good approximation to the integral of f on $[a, b]$ (see Fig. 4.3.9). Notice that the Riemann sum

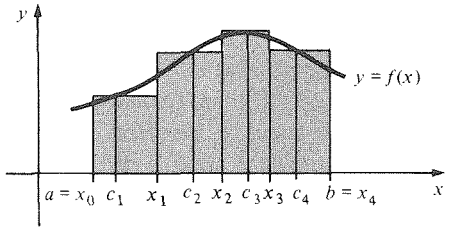

Figure 4.3.9. The area of the shaded region is a Riemann sum for f on $[a, b]$.

is formed by "sampling" the values of f at points $c_1, \ldots, c_n$, "weighting" the samples according to the lengths of the intervals from which the c_i's are chosen, and then adding.

If we choose a sequence of partitions, one for each n, such that the lengths Δx_i approach zero as n becomes larger, then the Riemann sums approach the integral $\int_a^b f(x)\,dx$ in the limit as $n \to \infty$. From this and Fig. 4.3.9, we again see the connection between integrals and areas.

Just as the derivatives may be defined as a limit of difference quotients, so the integral may be defined as a limit of Riemann sums; the integral as defined this way is called the *Riemann integral*.

[4] After the German mathematician Bernhard Riemann (1826–1866).

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Riemann Sums

Choose, for each n, a partition of $[a,b]$ into n subintervals such that the maximum of Δx_i in the nth partition approaches zero as $n \to \infty$. If c_i is a point chosen in the interval $[x_{i-1}, x_i]$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

Example 6 Write $\int_0^1 x^3\,dx$ as a limit of sums.

Solution As in Example 6, Section 4.2, divide $[0,1]$ into n equal parts by the partition $(0, 1/n, 2/n, \ldots, (n-1)/n, 1)$. Choose $c_i = i/n$, the right endpoint of the interval $[(i-1)/n, i/n]$. (We may choose any point we wish; the left endpoint or midpoint would have been just as good.) Then with $f(x) = x^3$, we get

$$S_n = \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} c_i^3 \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{i^3}{n^4}.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \int_0^1 x^3\,dx.$$

Thus, we can find $\int_0^1 x^3\,dx$ if we can evaluate this limit, or vice versa. ▲

Supplement to Section 4.3
Solar Energy

Besides the distance–velocity and area problems, which we used to introduce the integral, there are other physical problems that could be used in the same way. Here, we consider the problem of computing solar energy and shall see how it, too, leads naturally to the integral in terms of upper and lower sums.

Consider a solar cell attached to an energy storage unit (such as a battery) as in Fig. 4.3.10. When light shines on the solar cell, it is converted into electrical energy which is stored in the battery (as electrical–chemical energy) for later use.

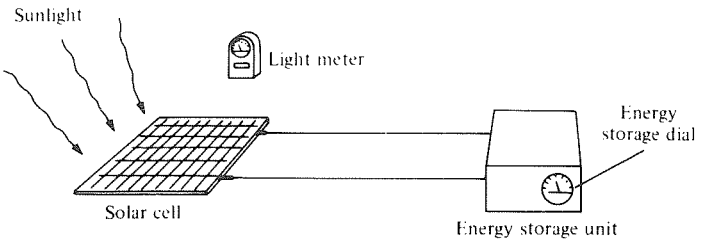

Figure 4.3.10. The storage unit accumulates the power received by the solar cell.

We will be interested in the relation between the amount E of energy stored and the intensity I of the sunlight. The number E can be read off a dial on the energy storage device; I can be measured with a photographer's light meter. (The units in which E and I are measured are unimportant for this discussion.)

Experiments show that when the solar cell is exposed to a steady source of sunlight, the change ΔE in the amount of energy stored is proportional to

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the product of the intensity I and the length Δt of the period of exposure. Thus

$$\Delta E = \kappa I \Delta t,$$

where κ is a constant depending on the apparatus and on the units used to measure energy, time, and intensity. (We can imagine κ being told to us as a manufacturer's specification.)

The intensity I can change—for example, the sun can move behind a cloud. If during two periods, Δt_1 and Δt_2, the intensity is, respectively, I_1 and I_2, then the total change in energy is the sum of the energies stored over each individual period. That is,

$$\Delta E = \kappa I_1 \Delta t_1 + \kappa I_2 \Delta t_2 = \kappa(I_1 \Delta t_1 + I_2 \Delta t_2).$$

Likewise, if there are n periods, $\Delta t_1, \ldots, \Delta t_n$, during which the intensity is $I_1, \ldots, I_n$ (as in Fig. 4.3.11(a)), the energy stored will be the sum of n terms,

$$\Delta E = \kappa(I_1 \Delta t_1 + I_2 \Delta t_2 + \cdots + I_n \Delta t_n) = \kappa \sum_{i=1}^{n} I_i \Delta t_i .$$

Notice that this sum is exactly κ times the integral of the step function g, where $g(t) = I_i$ on the interval of length Δt_i.

In practice, as the sun moves gradually behind the clouds and its elevation in the sky changes, the intensity I of sunlight does not change by jumps but varies continuously with t (Fig. 4.3.11(b)). The change in stored

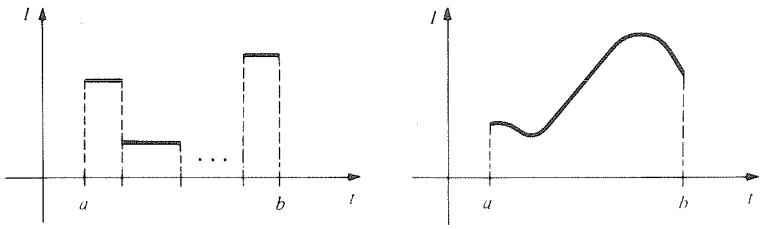

Figure 4.3.11. The intensity of sunlight varying with time.

energy ΔE can still be measured on the energy storage meter, but it can no longer be represented as a sum in the ordinary sense. In fact, the intensity now takes on infinitely many values, but it does not stay at a given value for any length of time.

If $I = f(t)$, the true change in stored energy is given by the integral

$$\Delta E = \kappa \int_a^b f(t)\, dt,$$

which is κ times the area under the graph $I = f(t)$. If $g(t)$ is a step function with $g(t) \leqslant f(t)$, then the integral of g is less than or equal to the integral of $f(t)$. This is in accordance with our intuition: the less the intensity, the less the energy stored.

The passage from step functions to general functions in the definition of the integral and the interpretation of the integral can be carried out in many contexts; this gives integral calculus a wide range of applications.

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Exercises for Section 4.3

In Exercises 1–4, draw a graph of the given step function, and compute the signed area of the region between its graph and the x axis.

1. $g(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1, \\ -3 & \text{if } 1 < x \leqslant 2. \end{cases}$

2. $g(x) = \begin{cases} -4 & \text{if } -1 \leqslant x < 0, \\ 2 & \text{if } 0 \leqslant x \leqslant 1, \\ 3 & \text{if } 1 < x \leqslant 2. \end{cases}$

3. $g(x) = \begin{cases} 1 & \text{if } -2 \leqslant x < -1, \\ -1 & \text{if } -1 \leqslant x \leqslant 0. \end{cases}$

4. $g(x) = \begin{cases} -3 & \text{if } -3 \leqslant x \leqslant -2, \\ -2 & \text{if } -2 < x \leqslant -1, \\ -1 & \text{if } -1 < x \leqslant 0. \end{cases}$

In Exercises 5–8, compute the indicated integrals.

5. $\int_0^2 g(x)\,dx$, g as in Exercise 1.

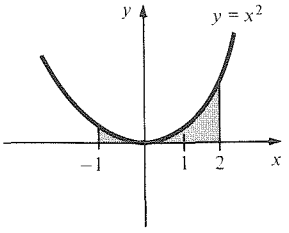
6. $\int_{-1}^0 g(x)\,dx$, g as in Exercise 2.

7. $\int_{-2}^0 g(x)\,dx$, g as in Exercise 3.

8. $\int_{-2}^0 g(x)\,dx$, g as in Exercise 4.

In Exercises 9–12, write the signed areas of the shaded regions in terms of integrals. (See Figure 4.3.12.)

9.



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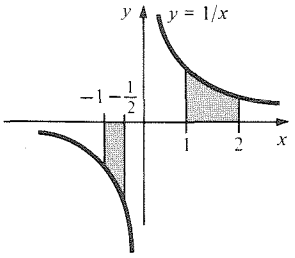

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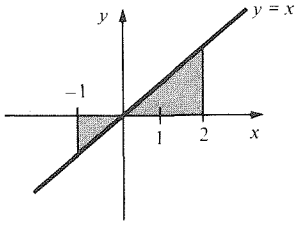

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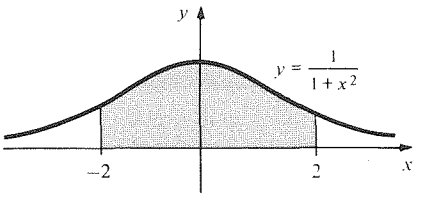

Figure 4.3.12. Graphs for Exercises 9–12.

13. Find $\int_2^4 (1/x)\,dx$ to within an error of no more than $\frac{1}{10}$.

14. If you used the method in Example 4 to calculate $\int_1^2 (1/x)\,dx$ to within $\frac{1}{100}$, how many subintervals would you need?

15. Estimate $\int_1^2 (1/x^2)\,dx$ to within $\frac{1}{10}$.

16. Estimate $\int_1^2 (1/x^2)\,dx$ to within $\frac{1}{100}$.

17. A bus moves on the line with velocity given by $v = 5(t^2 - 5t + 6)$. Write a formula in terms of integrals for:
    (a) the displacement of the bus between $t = 0$ and $t = 3$;
    (b) the actual distance the bus travels between $t = 0$ and $t = 3$.

18. A bus moves on the line with velocity given by $v = 6t^2 - 30t + 24$. Write a formula in terms of integrals for:
    (a) the displacement of the bus between $t = 0$ and $t = 5$;
    (b) the actual distance the bus travels between $t = 0$ and $t = 5$.

In Exercises 19–22, write the given integral as a limit of sums.

19. $\int_0^1 x^5\,dx$.

20. $\int_0^1 9x^3\,dx$.

21. $\int_2^4 \frac{1}{1+x^2}\,dx$.

22. $\int_3^4 \frac{x^2}{1+x}\,dx$.

23. Show that $-3 \leqslant \int_1^2 (t^3 - 4)\,dt \leqslant 4$.

24. Show that $\int_0^1 t^{10}\,dt \leqslant 1$.

25. Let $f(t)$ be defined by

$$f(t) = \begin{cases} 2 & \text{if } 0 \leqslant t < 1, \\ 0 & \text{if } 1 \leqslant t < 3, \\ -1 & \text{if } 3 \leqslant t \leqslant 4. \end{cases}$$

For any number x in $(0, 4]$, $f(t)$ is a step function on $[0, x]$.

(a) Find $\int_0^x f(t)\,dt$ as a function of x. (You will need to use different formulas on different intervals.)

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(b) Let $F(x) = \int_0^x f(t)\,dt$, for x in $(0, 4]$. Draw a graph of F.

(c) At which points is F differentiable? Find a formula for $F'(x)$.

26. Let f be the function defined by

$$f(x) = \begin{cases} 2, & 1 \leqslant x < 4, \\ 5, & 4 \leqslant x < 7, \\ 1, & 7 \leqslant x \leqslant 10. \end{cases}$$

(a) Find $\int_1^{10} f(x)\,dx$.

(b) Find $\int_2^9 f(x)\,dx$.

(c) Suppose that g is a function on $[1, 10]$ such that $g(x) \leqslant f(x)$ for all x in $[1, 10]$. What inequality can you derive for $\int_1^{10} g(x)\,dx$?

(d) With $g(x)$ as in part (c), what inequalities can you obtain for $\int_1^{10} 2g(x)\,dx$ and $\int_1^{10} -g(x)\,dx$? [*Hint*: Find functions like f with which you can compare $2g$ and $-g$.]

27. Let $f(t)$ be the "greatest integer function"; that is, $f(t)$ is the greatest integer which is less than or equal to t—for example, $f(n) = n$ for any integer, $f(5\frac{1}{2}) = 5$, $f(-5\frac{1}{2}) = -6$, and so on.

(a) Draw a graph of $f(t)$ on the interval $[-4, 4]$.

(b) Find $\int_0^1 f(t)\,dt$, $\int_0^6 f(t)\,dt$, $\int_{-2}^2 f(t)\,dt$, and $\int_0^{4.5} f(t)\,dt$.

(c) Find a general formula for $\int_0^n f(t)\,dt$, where n is any positive integer.

(d) Let $F(x) = \int_0^x f(t)\,dt$, where $x > 0$. Draw a graph of F for $x \in [0, 4]$, and find a formula for $F'(x)$, where it is defined.

28. A rod 1 meter long is made of 100 segments of equal length such that the linear density of the kth segment is $30k$ grams per meter. What is the total mass of the rod?

29. The volume of a rod of uniform shape is $A\,\Delta x$, where A is the cross-sectional area and Δx is the length.

(a) Suppose that the rod consists of n pieces, with the ith piece having cross-sectional area A_i and length Δx_i. Write a formula for the volume.

(b) Suppose that the cross-sectional area is $A = f(x)$, where f is a function on $[0, L]$, L being the total length of the rod. Write a formula for the volume of the rod, using the integral notation.

★30. Suppose that $f(x)$ is a step function on $[a, b]$, and let $g(x) = f(x) + k$, where k is a constant.

(a) Show that $g(x)$ is a step function.

(b) Find $\int_a^b g(x)\,dx$ in terms of $\int_a^b f(x)\,dx$.

★31. Let $h(x) = kf(x)$, where $f(x)$ is a step function on $[a, b]$.

(a) Show that $h(x)$ is a step function.

(b) Find $\int_a^b h(x)\,dx$ in terms of $\int_a^b f(x)\,dx$.

★32. For $x \in [0, 1]$ let $f(x)$ be the first digit after the decimal point in the decimal expansion of x.

(a) Draw a graph of f. (b) Find $\int_0^1 f(x)\,dx$.

★33. Define the functions f and g on $[0, 3]$ as follows:

$$f(x) = \begin{cases} 4, & 0 \leqslant x < 1, \\ -1, & 1 \leqslant x < 2, \\ 2, & 2 \leqslant x \leqslant 3, \end{cases}$$

$$g(x) = \begin{cases} 2, & 0 \leqslant x < 1\frac{1}{2}, \\ 1, & 1\frac{1}{2} \leqslant x \leqslant 3. \end{cases}$$

(a) Draw the graph of $f(x) + g(x)$ and compute $\int_0^3 [f(x) + g(x)]\,dx$.

(b) Compute $\int_1^2 [f(x) + g(x)]\,dx$.

(c) Compare $\int_0^3 2f(x)\,dx$ with $2\int_0^3 f(x)\,dx$.

(d) Show that

$$\int_0^3 [f(x) - g(x)]\,dx = \int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx.$$

(e) Is the following true?

$$\int_0^3 f(x) \cdot g(x)\,dx \stackrel{?}{=} \int_0^3 f(x)\,dx \cdot \int_0^3 g(x)\,dx.$$

★34. Suppose that f is a continuous function on $[a, b]$ and that $f(x) \neq 0$ for all x in $[a, b]$. Assume that $a \neq b$ and that $f((a + b)/2) = 1$. Prove that $\int_a^b f(x)\,dx > 0$. [*Hint*: Find a lower sum.]

★35. Compute the exact value of $\int_0^1 x^5\,dx$ by using Riemann sums and the formula

$$1^5 + 2^5 + 3^5 + \cdots + N^5 = \frac{N^6}{6} + \frac{N^5}{2} + \frac{5N^4}{12} - \frac{N^2}{12}.$$

★36. Let the function f be defined on $[0, 3]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number,} \\ 2 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Using the fact that between every two real numbers there lie both rationals and irrationals, show that every upper sum for f on $[0, 3]$ is at least 6.

(b) Show that every lower sum for f on $[0, 3]$ is at most 0.

(c) Is f integrable on $[0, 3]$? Explain.

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4.4 The Fundamental Theorem of Calculus

The processes of integration and differentiation are inverses to one another.

We now know two ways of expressing the solution of the distance–velocity problem. Let us recall the problem and these two ways.

Problem A bus moves on a straight line with given velocity $v = f(t)$ for $a \leqslant t \leqslant b$. Find the displacement Δd of the bus during this time interval.

First Solution The first solution uses antiderivatives and was presented in Section 2.5. Let $y = F(t)$ be the position of the bus at time t. Then since $v = dy/dt$, i.e., $f = F'$, F is an antiderivative of f. The displacement is the final position minus the initial position; i.e.,

$$\Delta d = F(b) - F(a), \tag{1}$$

the difference between the values of the antiderivative at $t = a$ and $t = b$.

Second Solution The second solution uses the integral as defined in the previous section. We saw that

$$\Delta d = \int_a^b f(t)\,dt. \tag{2}$$

We arrived at formulas (1) and (2) by rather different routes. However, the displacement must be the same in each case. Equating (1) and (2), we get

$$F(b) - F(a) = \int_a^b f(t)\,dt. \tag{3}$$

This equality is called the *fundamental theorem of calculus*. It expresses the integral in terms of an antiderivative and establishes the key link between differentiation and integration.

The argument by which we arrived at (3) was based on a physical model. Later, in this section, we shall also give a purely mathematical proof.

With a slight change of notation, we restate (3) in the following box.

Fundamental Theorem of Calculus

Suppose that the function F is differentiable everywhere on $[a, b]$ and that F' is integrable on $[a, b]$. Then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In other words, if f is integrable on $[a, b]$ and has an antiderivative F, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We may use this theorem to find the integral which we previously computed "by hand" (Example 6, Section 4.2).

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Example 1 Using the fundamental theorem of calculus, compute $\int_0^1 x\,dx$.

Solution By the power rule, an antiderivative for $f(x) = x$ is $F(x) = \frac{1}{2}x^2$. (You could also have found $F(x)$ by guessing, and you can always check the answer by differentiating $\frac{1}{2}x^2$.) The fundamental theorem gives

$$\int_0^1 x\,dx = \int_0^1 f(x)\,dx = F(1) - F(0) = \tfrac{1}{2}\cdot 1^2 - \tfrac{1}{2}\cdot 0^2 = \tfrac{1}{2},$$

which agrees with our earlier result. ▲

Next, we use the fundamental theorem to obtain a new result.

Example 2 Using the fundamental theorem of calculus, compute $\int_a^b x^2\,dx$.

Solution Let $f(x) = x^2$; again by the power rule, we may take $F(x) = \frac{1}{3}x^3$. By the fundamental theorem, we have

$$\int_a^b x^2\,dx = \int_a^b f(x)\,dx = F(b) - F(a) = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3.$$

We conclude that $\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$. This gives the area under a segment of the parabola $y = x^2$ (Fig. 4.4.1). ▲

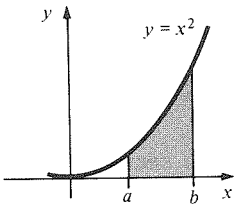

Figure 4.4.1. The shaded area is $\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$.

We can summarize the integration method provided by the fundamental theorem as follows:

Fundamental Integration Method

To integrate the function $f(x)$ over the interval $[a,b]$: find an antiderivative $F(x)$ for $f(x)$, then evaluate F at a and b and subtract the results:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Notice that the fundamental theorem does not specify which antiderivative to use. However, if F_1 and F_2 are two antiderivatives of f on $[a,b]$, they differ by a constant (see Section 3.6); $F_1(t) = F_2(t) + C$, and so

$$F_1(b) - F_1(a) = \big[F_2(b) + C\big] - \big[F_2(a) + C\big] = F_2(b) - F_2(a).$$

(The C's cancel.) Thus all choices of F give the same result.

Expressions of the form $F(b) - F(a)$ occur so often that it is useful to have a special notation for them.

Notation for the Fundamental Theorem

$$F(x)\Big|_a^b \quad \text{means} \quad F(b) - F(a).$$

Example 3 Find $(x^3 + 5)\big|_2^3$.

Solution Here $F(x) = x^3 + 5$ and

$$\begin{aligned}(x^3 + 5)\big|_2^3 &= F(3) - F(2)\\ &= (3^3 + 5) - (2^3 + 5)\\ &= 32 - 13 = 19.\ \blacktriangle\end{aligned}$$

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In terms of this new notation, we can write the formula of the fundamental theorem of calculus in the form

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b,$$

where F is an antiderivative of f on $[a,b]$.

Example 4 Find $\int_2^6 (x^2+1)\,dx$.

Solution By the sum and power rules for antiderivatives, and antiderivative for x^2+1 is $\frac{1}{3}x^3 + x$. By the fundamental theorem,

$$\begin{aligned}\int_2^6 (x^2+1)\,dx &= \left(\frac{1}{3}x^3 + x\right)\Big|_2^6 \\ &= \left(\frac{6^3}{3}+6\right) - \left(\frac{2^3}{3}+2\right) \\ &= 78 - 4\tfrac{2}{3} = 73\tfrac{1}{3}. \blacktriangle\end{aligned}$$

Example 5 Evaluate $\int_1^2 \frac{1}{x^4}\,dx$.

Solution An antiderivative of $1/x^4 = x^{-4}$ is $-1/3x^3$, since

$$\frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = -\frac{1}{3}\cdot(-3)x^{-4} = x^{-4}.$$

Hence

$$\int_1^2 \frac{1}{x^4}\,dx = -\frac{1}{3x^3}\Big|_1^2 = \left(-\frac{1}{3\cdot 2^3}\right) - \left(-\frac{1}{3\cdot 1^3}\right) = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}. \blacktriangle$$

We will now give a complete proof of the fundamental theorem of calculus. The basic idea is as follows: letting F be an antiderivative for f on $[a,b]$, we will show that the number $F(b) - F(a)$ lies between any lower and upper sums for f on $[a,b]$. Since f is assumed integrable, it has upper and lower sums arbitrarily close together, and the only number with this property is the integral of f (see page 217). Thus, we will have $F(b) - F(a) = \int_a^b f(x)\,dx$.

Proof of the Fundamental Theorem For the lower sums, we must show that any step function g below f on (a,b) has integral at most $F(b) - F(a)$. So let $k_1, k_2, \ldots, k_n$ be the values of g on the partition intervals $(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n)$ (See Fig. 4.4.2). On

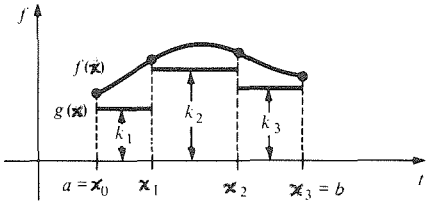

Figure 4.4.2. The integral of g is a lower sum for f on $[a,b]$.

(x_{i-1}, x_i), $k_i \leq f(x) = F'(x)$, so the difference quotient for F satisfies the inequality $k_i \leq [F(x_i) - F(x_{i-1})]/[x_i - x_{i-1}]$, by the first consequence of the mean value theorem (Section 3.6). Thus, $k_i \Delta x_i \leq F(x_i) - F(x_{i-1})$. Summing from $i = 1$ to n, we get

$$\sum_{i=1}^n k_i \Delta x_i \leq \left[F(x_1) - F(x_0)\right] + \left[F(x_2) - F(x_1)\right] + \cdots + \left[F(x_n) - F(x_{n-1})\right].$$

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The left-hand side is just the integral of g on $[a,b]$, while the right-hand side is a telescoping sum which collapses to $F(x_n) - F(x_0)$; so we have proven that $\int_a^b g(x)\,dx \leqslant F(b) - F(a)$.

An identical argument works for upper sums: If h is a step function above f on (a,b), then $F(b) - F(a) \leqslant \int_a^b h(x)\,dx$ (see Exercise 49). Thus the proof of the fundamental theorem is complete. ■

Here are two more examples illustrating the use of the fundamental theorem. Notice that any letter can be used as the variable of integration, just like the "dummy variable" in summation.

Example 6 Find $\int_0^4 (t^2 + 3t^{7/2})\,dt$.

Solution By the sum, constant multiple, and power rules for antiderivatives, an antiderivative for $t^2 + 3t^{7/2}$ is $(t^3/3) + 3 \cdot (2/9)t^{9/2}$. Thus,

$$\int_0^4 (t^2 + 3t^{7/2})\,dt = \left(\frac{t^3}{3} + \frac{2t^{9/2}}{3} \right)\Big|_0^4$$

$$= \frac{4^3}{3} + \frac{2 \cdot 2^9}{3} = \frac{1088}{3} . \blacktriangle$$

In the next example, some algebraic manipulations are needed before the integral is computed.

Example 7 Compute $\int_1^2 \frac{(s+5)^2}{s^4}\,ds$.

Solution The integrand may be broken apart:

$$\frac{(s+5)^2}{s^4} = \frac{s^2 + 10s + 25}{s^4} = \frac{1}{s^2} + \frac{10}{s^3} + \frac{25}{s^4} .$$

We can find an antiderivative term by term, by the power rule:

$$\int_1^2 \left(\frac{1}{s^2} + \frac{10}{s^3} + \frac{25}{s^4} \right) ds = \int_1^2 (s^{-2} + 10s^{-3} + 25s^{-4})\,ds$$

$$= \left(\frac{s^{-1}}{(-1)} + \frac{10s^{-2}}{(-2)} + \frac{25s^{-3}}{(-3)} \right)\Big|_1^2$$

$$= \left(-\frac{1}{s} - \frac{10}{2s^2} - \frac{25}{3s^3} \right)\Big|_1^2$$

$$= -\left(\frac{1}{s} + \frac{10}{2s^2} + \frac{25}{3s^3} \right)\Big|_1^2$$

$$= -\left(\left(\frac{1}{2} + \frac{5}{4} + \frac{25}{3 \cdot 8} \right) - \left(1 + 5 + \frac{25}{3} \right) \right)$$

$$= -\left(\frac{67}{24} - \frac{43}{3} \right) = \frac{277}{24} \approx 11.54. \blacktriangle$$

Next we use the fundamental theorem to solve area and distance–velocity problems. Let us first recall, from Sections 4.2 and 4.3, the situation for areas under graphs.

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Area under a Graph

If $f(x) \geqslant 0$ for x in $[a,b]$, the area under the graph of f between $x = a$ and $x = b$ is

$$\int_a^b f(x)\,dx.$$

If f is negative at some points of $[a,b]$, then $\int_a^b f(x)\,dx$ is the signed area of the region between the graph of f, the x axis, and the lines $x = a$ and $x = b$.

Example 8 (a) Find the area of the region bounded by the x axis, the y axis, the line $x = 2$, and the parabola $y = x^2$. (b) Compute the area of the region shown in Fig. 4.4.3.

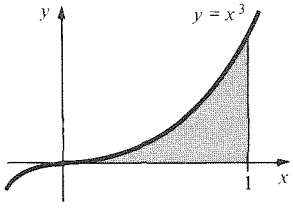

Figure 4.4.3. Compute this area.

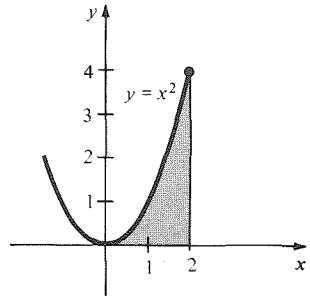

Figure 4.4.4. The shaded area equals $\int_0^2 x^2\,dx$.

Solution (a) The region described is that under the graph of $f(x) = x^2$ on $[0,2]$ (Fig. 4.4.4). The area of the region is $\int_0^2 x^2\,dx = \frac{1}{3}x^3\big|_0^2 = \frac{8}{3}$.
(b) The region is that under the graph of $y = x^3$ from $x = 0$ to $x = 1$, so its area is $\int_0^1 x^3\,dx$. By the fundamental theorem,

$$\int_0^1 x^3\,dx = \frac{x^4}{4}\bigg|_0^1 = \frac{1}{4}.$$

Thus, the area is $\frac{1}{4}$. ▲

Example 9 (a) Interpret $\int_0^2 (x^2 - 1)\,dx$ in terms of areas and evaluate. (b) Find the shaded area in Figure 4.4.5.

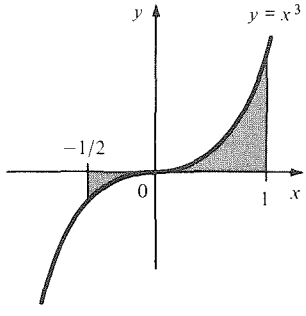

Figure 4.4.5. Find the area of this region.

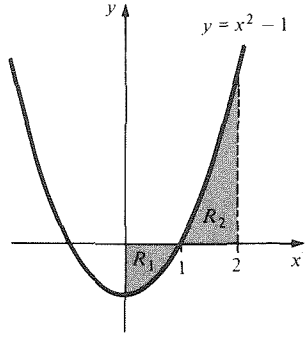

Figure 4.4.6. $\int_0^2 (x^2 - 1)\,dx$ is the difference between the areas of R_2 and R_1.

Solution (a) Refer to Fig. 4.4.6. We know that the integral represents the signed area of the region between the graph of $y = x^2 - 1$ and the x axis. In other words, it is

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the area of R_2 minus the area of R_1. Evaluating,

$$\int_0^2 (x^2 - 1)\,dx = \left(\frac{x^3}{3} - x \right)\Big|_0^2 = \frac{8}{3} - 2 = \frac{2}{3}.$$

(b) For functions which are negative on part of an interval, we must recall, from Section 4.3, that the integral represents the *signed* area between the graph and the x axis. To get the ordinary area, we must integrate piece by piece.

The area from $x = 0$ to $x = 1$ is $\int_0^1 x^3\,dx$. The negative of the area from $x = -\frac{1}{2}$ to $x = 0$ is $\int_{-1/2}^0 x^3\,dx$. Thus the total area is

$$\begin{aligned} A &= -\int_{-1/2}^0 x^3\,dx + \int_0^1 x^3\,dx \\ &= -\frac{x^4}{4}\Big|_{-1/2}^0 + \frac{x^4}{4}\Big|_0^1 = \frac{(1/2)^4}{4} + \frac{1}{4} \\ &= \frac{1}{4 \cdot 16} + \frac{1}{4} = \frac{17}{64}. \blacktriangle \end{aligned}$$

Finally, in this section, we consider the use of the fundamental theorem to solve displacement problems. The following box summarizes the method, which was justified earlier in this section.

Displacements and Velocity

If a particle on the x axis has velocity $v = f(t)$ and position $x = F(t)$, then the displacement $F(b) - F(a)$ between the times $t = a$ and $t = b$ is obtained by integrating the velocity from $t = a$ to $t = b$:

$$\begin{pmatrix} \text{Displacement from} \\ \text{time } t = a \text{ to } t = b \end{pmatrix} = \int_a^b (\text{velocity})\,dt.$$

Example 10 An object moving in a straight line has velocity $v = 5t^4 + 3t^2$ at time t. How far does the object travel between $t = 1$ and $t = 2$?

Solution The displacement equals the total distance travelled in this case, since $v > 0$. Thus, the displacement is

$$\Delta d = \int_1^2 (5t^4 + 3t^2)\,dt = (t^5 + t^3)\Big|_1^2 = (32 + 8) - (1 + 1) = 38.$$

Thus, the object travels 38 units of length between $t = 1$ and $t = 2$. ▲

We have seen that the geometric interpretation of integrals of functions that can sometimes be negative requires the notion of signed area. Likewise, when velocities are negative, we have to be careful with signs. The integral is always the displacement; to get the actual distance travelled, we must change the sign of the integral over the periods when the velocity is negative. See Fig. 4.4.7 for a typical situation.

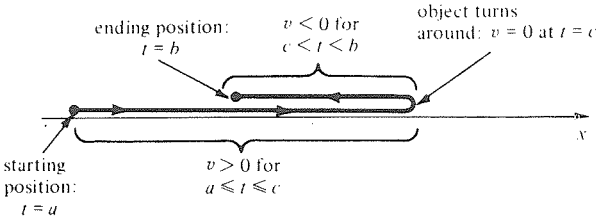

Figure 4.4.7. The total distance travelled is $\int_a^c v\,dt - \int_c^b v\,dt$; the displacement is $\int_a^b v\,dt$.

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Example 11 An object on the x axis has velocity $v = 2t - t^2$ at time t. If it starts out at $x = -1$ at time $t = 0$, where is it at time $t = 3$? How far has it travelled?

Solution Let $x = f(t)$ be the position at time t. Then

$$f(3) - f(0) = \int_0^3 (2t - t^2)\,dt$$

$$= \left(t^2 - \frac{t^3}{3}\right)\Big|_0^3 = 9 - \frac{27}{3} = 0.$$

Since $f(0) = -1$, the object is again at $x = 0 + f(0) = -1$ at time $t = 3$.

The object turns around when v changes sign, namely, at those t where $2t - t^2 = 0$ or $t = 0, 2$. For $0 < t < 2$, $v > 0$, and for $2 < t < 3$, $v < 0$. The total distance travelled is therefore

$$\int_0^2 (2t - t^2)\,dt - \int_2^3 (2t - t^2)\,dt$$

$$= \left(t^2 - \frac{t^3}{3}\right)\Big|_0^2 - \left(t^2 - \frac{t^3}{3}\right)\Big|_2^3$$

$$= \left(4 - \frac{8}{3}\right) - \left(9 - \frac{27}{3}\right) + \left(4 - \frac{8}{3}\right)$$

$$= \frac{8}{3}. \blacktriangle$$

Exercises for Section 4.4

Using the fundamental theorem of calculus, compute the integrals in Exercises 1–4.

1. $\int_1^3 x^3\,dx$.
2. $\int_2^3 x^2\,dx$.
3. $\int_4^6 3x\,dx$.
4. $\int_1^8 (1 + \sqrt{x})\,dx$.

Compute the quantities in Exercises 5–8.

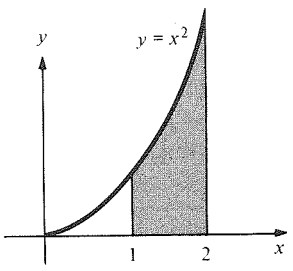
5. $x^{3/4}|_0^2$.
6. $(x^2 + 2\sqrt[3]{x})|_0^8$.
7. $(3x^2 + 5)|_1^3$.
8. $(x^4 + x^2 + 2)|_{-2}^2$.

Evaluate the integrals in Exercises 9–24.

9. $\int_a^b s^{4/3}\,ds$.
10. $\int_{-1}^2 (t^4 + 8t)\,dt$.
11. $\int_1^2 4\pi r^{2/3}\,dr$.
12. $\int_{-1}^1 (t^4 + t^{917})\,dt$.
13. $\int_0^{10} \left(\frac{t^4}{100} - t^2\right) dt$.
14. $\int_{-4}^0 (1 + x^2 - x^3)\,dx$.
15. $\int_{-1}^2 (1 + t^2)^2\,dt$.
16. $\int_1^2 \left(s^3 + \frac{1}{s^2}\right) ds$.
17. $\int_1^2 \frac{dt}{(t+4)^3}$.
18. $\int_{\pi/2}^{\pi} (3 + z^2)\,dz$.
19. $\int_1^2 \frac{(1+t^2)^2}{t^2}\,dt$.
20. $\int_1^2 \frac{t^2 + 8t + 1}{t^4}\,dt$.
21. $\int_1^2 \frac{(x^2+5)^2}{x^4}\,dx$.
22. $\int_{-2}^{-1} \frac{(x^2+x)^2}{x}\,dx$.
23. $\int_2^3 \frac{u^3 - 1}{u - 1}\,du$.
24. $\int_2^4 \frac{u^4 - 1}{u - 1}\,du$.

Calculate the areas of the regions in Exercises 25–28 (Figure 4.4.8).

25.



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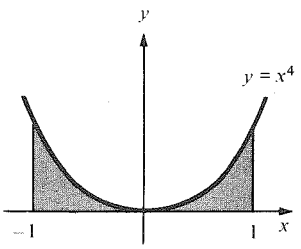

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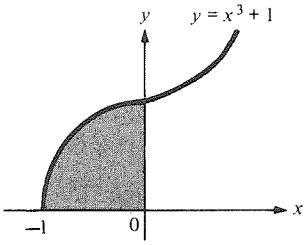

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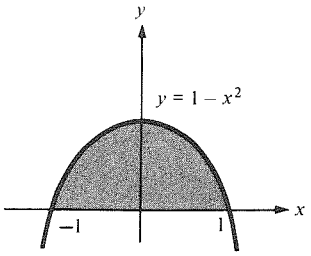

Figure 4.4.8. Regions for Exercises 25–28.

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Interpret the integrals in Exercises 29 and 30 in terms of areas, sketch, and evaluate.

29. $\int_0^2 (x^3 - 1)\,dx$.

30. $\int_1^2 (x^2 - 3)\,dx$.

In Exercises 31–40, find the area of the region between the graph of each of the following functions and the x axis on the given interval and sketch.

31. x^3 on $[0, 2]$.
32. $1/x^2$ on $[1, 2]$.
33. $x^2 + 2x + 3$ on $[1, 2]$.
34. $x^3 + 3x + 2$ on $[0, 2]$.
35. $x^4 + 2$ on $[-1, 1]$.
36. $3x^4 - 2x + 2$ on $[-1, 1]$.
37. $x^4 + 3x^2 + 1$; $-2 \leqslant x \leqslant 1$.
38. $8x^6 + 3x^4 - 2$; $1 \leqslant x \leqslant 2$.
39. $(1/x^3) + x^2$; $1 \leqslant x \leqslant 3$.
40. $(3x + 5)/x^3$; $1 \leqslant x \leqslant 2$.

41. An object moving in a straight line has velocity $v = 6t^4 + 3t^2$ at time t. How far does the object travel between $t = 1$ and $t = 10$?
42. An object moving in a straight line has velocity $v = 2t^3 + t^4$ at time t. How far does the object travel between $t = 0$ and $t = 2$?
43. The velocity of an object on the x axis is $v = 4t - 2t^2$. If it is at $x = 1$ at $t = 0$, where is it at $t = 4$? How far has it travelled?
44. The velocity of an object on the x axis is $v = t^2 - 3t + 2$. If the object is at $x = -1$ at $t = 0$, where is it at $t = 2$? How far has it travelled?
45. The velocity of a stone dropped from a balloon is $32t$ feet per second, where t is the time in seconds after release. How far does the stone travel in the first 10 seconds?
46. How far does the stone in Exercise 45 travel in the second 10 seconds after its release? The third 10 seconds?

★47. An object is thrown upwards from the earth's surface with a velocity v_0. (a) How far has it travelled after it returns? (b) How far has it travelled when its velocity is $-\frac{1}{2}v_0$?

★48. Suppose that F is continuous on $[0, 2]$, that $F'(x) < 2$ for $0 \leqslant x \leqslant \frac{1}{3}$, and that $F'(x) < 1$ whenever $\frac{1}{3} \leqslant x \leqslant 2$. What can you say about the difference $F(2) - F(0)$?

★49. Prove that if $h(t)$ is a step function on $[a, b]$ such that $f(t) \leqslant h(t)$ for all t in the interval (a, b), then $F(b) - F(a) \leqslant \int_a^b h(t)\,dt$, where F is any antiderivative for f on $[a, b]$.

★50. Let $a_0, \ldots, a_n$ be a given set of numbers and $\delta_i = a_i - a_{i-1}$. Let $b_k = \sum_{i=1}^{k} \delta_i$, $d_i = b_i - b_{i-1}$. Express the b's in terms of the a's and the d's in terms of the δ's.

4.5 Definite and Indefinite Integrals

Integrals and sums have similar properties.

When we studied antiderivatives in Section 2.5, we used the notation $\int f(x)\,dx$ for an antiderivative of f, and we called it an *indefinite integral*. This notation and terminology are consistent with the fundamental theorem of calculus. We can rewrite the fundamental theorem in terms of the indefinite integral in the following way.

Definite and Indefinite Integrals

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b$$

Notice that although the indefinite integral is a function involving an arbitrary constant, the expression

$$\left(\int f(x)\,dx\right)\Big|_a^b$$

represents a well-defined number, since the constant cancels when we subtract the value at a from the value at b.

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An expression of the form $\int_a^b f(x)\,dx$ with the endpoints specified, which we have been calling simply "an integral," is sometimes called a *definite* integral to distinguish it from an indefinite integral.

Note that a definite integral is a number, while an indefinite integral is a *function* (determined up to an additive constant).

Remember that one may check an indefinite integral formula by differentiating.

Indefinite Integral Test

To check a given formula $\int f(x)\,dx = F(x) + C$, differentiate the right-hand side and see if you get the integrand $f(x)$.

Example 1 Check the formula $\int 3x^8\,dx = x^9/3 + C$.

Solution We differentiate the right-hand side using the power rule:

$$\frac{d}{dx}\left(\frac{x^9}{3} + C\right) = \frac{9x^8}{3} = 3x^8;$$

so the formula checks. ▲

The next example involves an integral that cannot be readily found with the antidifferentiation rules.

Example 2 (a) Check the formula $\int x(1+x)^6\,dx = \frac{1}{56}(7x-1)(1+x)^7 + C$. (Do not attempt to derive the formula.)

(b) Find $\int_0^2 x(1+x)^6\,dx$.

Solution (a) We differentiate the right-hand side using the product rule and power of a function rule:

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{56}(7x-1)(1+x)^7\right] &= \frac{1}{56}\left[7(1+x)^7 + (7x-1)7(1+x)^6\right] \\ &= \frac{1}{56}(1+x)^6\left[7(1+x) + 7(7x-1)\right] \\ &= (1+x)^6 x.\end{aligned}$$

Thus the formula checks.

(b) By the fundamental theorem and the formula we just checked, we have

$$\begin{aligned}\int_0^2 x(1+x)^6\,dx &= \frac{1}{56}(7x-1)(1+x)^7\Big|_0^2 \\ &= \frac{1}{56}\left[13\cdot 3^7 - (-1)\right] \\ &= \frac{28{,}432}{56} = \frac{3554}{7} \approx 507.7\end{aligned}$$ ▲

In the box on page 204 we listed five key properties of the summation process. In the following box we list the corresponding properties of the definite integral.

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Properties of the Definite Integral

1. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (sum rule).
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c a constant (constant multiple rule).
3. If $a < b < c$, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
4. If $f(x) = C$ is constant, then $\int_a^b f(x)\,dx = C(b - a)$.
5. If $f(x) \leqslant g(x)$ for all x satisfying $a \leqslant x \leqslant b$, then
$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$$

These properties hold for all functions f and g that have integrals. However, while it is technically a bit less general, it is much easier to deduce the properties from the antidifferentiation rules and the fundamental theorem of calculus, assuming not only that f and g have integrals, but that they have antiderivatives as well.

Example 3 Prove property 1 in the display above (assuming that f and g have antiderivatives).

Solution Let F be an antiderivative for f and G be one for g. Then $F + G$ is an antiderivative for $f + g$ by the sum rule for antiderivatives. Thus,

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= [F(x) + G(x)]\Big|_a^b \\ &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \ \blacktriangle\end{aligned}$$

Example 4 Prove property 5.

Solution If $f(x) \leqslant g(x)$ on (a, b), then $(F - G)'(x) = F'(x) - G'(x) = f(x) - g(x) \leqslant 0$ for x in (a, b). Since a function with a negative derivative is decreasing, we get

$$[F(b) - G(b)] - [F(a) - G(a)] \leqslant 0,$$

and so $F(b) - F(a) \leqslant G(b) - G(a)$. By the fundamental theorem of calculus, the last inequality can be written

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$$

as required. ▲

Properties 2 and 3 can be proved in a way similar to property 1. Note that property 4 is obvious, since we know how to compute areas of rectangles.

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Example 5 Explain property 3 in terms of (a) areas (assume that f is a positive function) and (b) distances and velocities.

Solution (a) Since $\int_a^c f(x)\,dx$ is the area under the graph of f from $x = a$ to $x = c$, property 3 merely states that the sum of the areas of regions A and B in Fig. 4.5.1 is the total area.

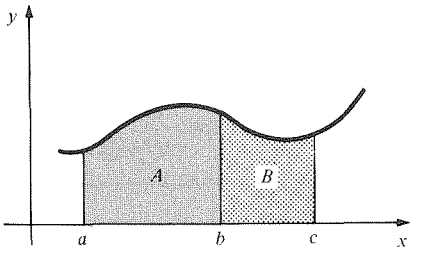

Figure 4.5.1. Illustrating the rule $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

(b) Property 3 states that the displacement for a moving object between times a and c equals the sum of the displacements between a and b and between b and c. ▲

We have defined the integral $\int_a^b f(x)\,dx$ when a is less than b; however, the right-hand side of the equation

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

makes sense even when $a \geqslant b$. Can we define $\int_a^b f(x)\,dx$ for the case $a \geqslant b$ so that this equation will still be true? The answer is simple:

If $b < a$ and f is integrable on $[b, a]$, we define

$$\int_a^b f(x)\,dx \quad \text{to be} \quad -\int_b^a f(x)\,dx.$$

If $a = b$, we define $\int_a^b f(x)\,dx$ to be zero.

Notice that if F' is integrable on $[b, a]$, where $b < a$, then by the preceding definition and the fundamental theorem,

$$\int_a^b F'(x)\,dx = -\int_b^a F'(x)\,dx = -\big[F(a) - F(b)\big] = F(b) - F(a),$$

so the equation $\int_a^b F'(x)\,dx = F(b) - F(a)$ is still valid.

"Wrong-Way" Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$;
2. $\int_a^a f(x)\,dx = 0$;
3. $\int_a^b F'(x)\,dx = F(b) - F(a)$, for all a and b.

Example 6 Find $\int_6^2 x^3\,dx$.

Solution $\int_6^2 x^3\,dx = (x^4/4)\big|_6^2 = \frac{1}{4}(16 - 1296) = -320$. (Although the function $f(x) = x^3$ is positive, the integral is negative. To explain this, we remark that as x goes from 6 to 2, "dx is negative.") ▲

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We have seen that the fundamental theorem of calculus enables us to compute integrals by using antiderivatives. The relationship between integration and differentiation is completed by an alternative version of the fundamental theorem. Let us first state and prove it; its geometric meaning will be given shortly.

Fundamental Theorem of Calculus: Alternative Version

If f is continuous on $[a,x]$, then $\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$.

We now justify the alternative version of the fundamental theorem. In Exercises 49–53, it is shown that f has an antiderivative F. Let us accept this fact here.

The fundamental theorem applied to f on the interval $[a, x]$ gives

$$\int_a^x f(s)\,ds = F(x) - F(a).$$

Differentiating both sides,

$$\begin{aligned}\frac{d}{dx}\int_a^x f(s)\,ds &= \frac{d}{dx}\left[F(x) - F(a)\right] \\ &= \frac{d}{dx}F(x) \qquad \text{(since } F(a) \text{ is constant)} \\ &= f(x) \qquad \text{(since } F \text{ is an antiderivative of } f\text{)}.\end{aligned}$$

Thus the alternative version is proved.

Notice that in the statement of the theorem we have changed the (dummy) variable of integration to the letter "s" to avoid confusion with the endpoint "x."

Example 7 Verify the formula $\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$ for $f(x) = x$.

Solution The integral in question is

$$\int_a^x f(s)\,ds = \int_a^x s\,ds = \frac{s^2}{2}\Big|_a^x = \frac{x^2}{2} - \frac{a^2}{2}.$$

Thus, $\frac{d}{dx}\left(\int_a^x f(s)\,ds\right) = \frac{d}{dx}\left(\frac{x^2}{2} - \frac{a^2}{2}\right) = x = f(x)$,

so the formula holds. ▲

Example 8 Let $F(x) = \int_2^x \frac{1}{1 + s^2 + s^3}\,ds$. Find $F'(3)$.

Solution Using the alternative version of the fundamental theorem, with $f(s) = 1/(1 + s^2 + s^3)$, we have $F'(3) = f(3) = 1/(1 + 3^2 + 3^3) = \frac{1}{37}$. Notice that we did not need to differentiate or integrate $1/(1 + s^2 + s^3)$ to get the answer. ▲

At the top of the next page, we summarize the two forms of the fundamental theorem.

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Fundamental Theorem of Calculus

Usual Version: $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Integrating the derivative of F gives the change in F.

Alternative Version: $\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$.

Differentiating the integral of f with respect to the upper limit gives f.

The alternative form of the fundamental theorem of calculus has an illuminating interpretation and explanation in terms of areas. Suppose that $f(x)$ is non-negative on $[a,b]$. Imagine uncovering the graph of f by moving a screen to the right, as in Fig. 4.5.2. When the screen is at x, the exposed area is

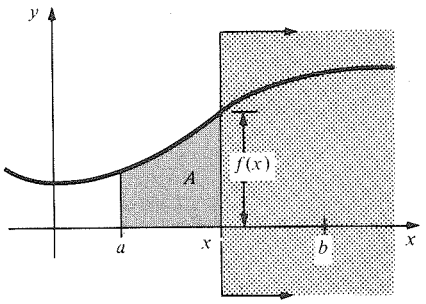

Figure 4.5.2. $dA/dx = f(x)$.

$A = \int_a^x f(s)\,ds$. The alternative version of the fundamental theorem can be phrased as follows: *as the screen moves to the right, the rate of change of exposed area A with respect to x, dA/dx, equals $f(x)$*. This same conclusion can be seen graphically by investigating the difference quotient,

$$\frac{A(x+\Delta x) - A(x)}{\Delta x}.$$

The quantity $A(x+\Delta x) - A(x)$ is the area under the graph of f *between* x and $x+\Delta x$. For Δx small, this area is approximately the area of the rectangle

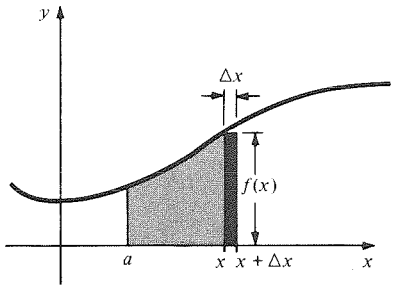

Figure 4.5.3. The geometry needed to explain why $dA/dx = f(x)$.

with base Δx and height $f(x)$, as in Fig. 4.5.3. Therefore,

$$\frac{A(x+\Delta x) - A(x)}{\Delta x} \approx \frac{f(x)\Delta x}{\Delta x} = f(x),$$

and the approximation gets better as Δx becomes smaller. Thus

$$\frac{A(x+\Delta x) - A(x)}{\Delta x}$$

approaches $f(x)$ as $\Delta x \to 0$, which means that $dA/dx = f(x)$. If f is continu-

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ous, this argument is the basis for a rigorous proof of the alternative version of the fundamental theorem. See Exercises 49–53 for additional details.

Exercises for Section 4.5

In Exercises 1–4, check the integration formula by differentiating the right-hand side.

1. $\int 5x^4\,dx = x^5 + C.$
2. $\int \frac{1+t}{t^3}\,dt = -\frac{1}{2t^2} - \frac{1}{t} + C.$
3. $\int 5(t^9 + t^4)\,dt = \frac{t^{10}}{2} + t^5 + C.$
4. $\int \frac{x - x^3 + 1}{x^3}\,dx = -\frac{1}{x} - x - \frac{1}{2x^2} + C.$
5. (a) Check the following integral
$$\int \frac{3t^2}{(1+t^3)^2}\,dt = \frac{t^3}{1+t^3} + C.$$
(b) Evaluate $\int_0^1 \frac{3t^2}{(1+t^3)^2}\,dt.$
6. (a) Check the following integration formula:
$$\int \frac{x^3 + 2x + 1}{(1-x)^5}\,dx = \frac{1}{x-1} + \frac{3}{2(x-1)^2} + \frac{5}{3(x-1)^3} + \frac{1}{(x-1)^4} + C$$
(b) Evaluate $\int_2^3 [(s^3 + 2s + 1)/(1-s)^5]\,ds.$
7. (a) Calculate the derivative of $\frac{x^3}{x^2+1}$.
(b) Find $\int_0^1 \frac{(3x^2 + x^4)}{(1+x^2)^2}\,dx.$
8. (a) Differentiate $\frac{x}{1+x}$ and $-\frac{1}{1+x}$.
(b) Find $\int_3^2 \frac{1}{(1+x)^2}\,dx$ in two ways.

Calculate the definite integrals in Exercises 9–18.

9. $\int_{-2}^3 (x^4 + 5x^2 + 2x + 1)\,dx.$
10. $\int_{-1}^1 (x^3 + 7)\,dx.$
11. $\int_{-2}^4 x^6\,dx.$
12. $\int_{-3}^{472} 0\,dt.$
13. $\int_1^2 \frac{x^2 + 2x + 2}{x^4}\,dx.$
14. $\int_1^8 \frac{1 + \theta^2}{\theta^4}\,d\theta.$
15. $\int_2^3 \frac{dt}{t^2}.$
16. $\int_2^{-2} t^4\,dt.$
17. $\int_1^2 (1 + 2t)^5\,dt.$
18. $\int_2^1 (1-x)^6\,dx.$
19. Explain property 2 of integration in terms of (a) areas and (b) distances and velocities.
20. Explain property 5 of integration in terms of (a) areas and (b) distances and velocities.

If $\int_0^1 f(x)\,dx = 3$, $\int_1^2 f(x)\,dx = 4$, and $\int_2^3 f(x)\,dx = -8$, calculate the quantities in Exercises 21–24, using the properties of integration.

21. $\int_0^2 f(x)\,dx.$
22. $\int_0^1 3f(x)\,dx.$
23. $\int_0^3 8f(x)\,dx.$
24. $\int_1^3 10f(x)\,dx.$

Calculate the integrals in Exercises 25–28.

25. $\int_3^2 x\,dx.$
26. $\int_8^4 (x^2 - 1)\,dx.$
27. $\int_{10}^9 \frac{x+1}{x^3}\,dx.$
28. $\int_{-3}^{-2} \frac{x^3 - 1}{x - 1}\,dx.$

Verify the formula $\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$ for the functions in Exercises 29 and 30.

29. $f(x) = x^3 - 1.$
30. $f(x) = x^4 - x^2 + x.$
31. Let $F(t) = \int_3^t \frac{1}{\left[(4-s)^2 + 8\right]^3}\,ds$. Find $F'(4)$.
32. Find $\frac{d}{dx}\int_0^x \frac{t^4}{1+t^6}\,dt.$

Evaluate the derivatives in Exercises 33–36.

33. $\frac{d}{dt}\int_0^t \frac{3}{(x^4 + x^3 + 1)^6}\,dx.$
34. $\frac{d}{dt}\int_3^t \frac{1}{x^4 + x^6}\,dx.$
35. $\frac{d}{dt}\int_t^3 x^2(1+x)^5\,dx.$
36. $\frac{d}{dt}\int_t^4 \frac{u^4}{(u^2+1)^3}\,du.$
37. Let $v(t)$ be the velocity of a moving object. In this context, interpret the formula
$$\frac{d}{dt}\int_a^t v(s)\,ds = v(t).$$
38. Interpret the alternative version of the fundamental theorem of calculus in the context of the solar energy example in the Supplement to Section 4.3.

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39. Suppose that

$$f(t) = \begin{cases} t^2, & 0 \leq t \leq 1, \\ 1, & 1 \leq t < 5, \\ (t-6)^2, & 5 \leq t \leq 6. \end{cases}$$

(a) Draw a graph of f on the interval $[0, 6]$.

(b) Find $\int_0^6 f(t)\,dt$.

(c) Find $\int_0^6 f(x)\,dx$.

(d) Let $F(t) = \int_0^t f(s)\,ds$. Find the formula for $F(t)$ in $[0, 6]$ and draw a graph of F.

(e) Find $F'(t)$ for t in $(0, 6)$.

40. (a) Give a formula for a function f whose graph is the broken line segment $ABCD$ in Fig. 4.5.4.

(b) Find $\int_3^{10} f(t)\,dt$.

(c) Find the area of quadrilateral $ABCD$ by means of geometry and compare the result with the integral in part (b).

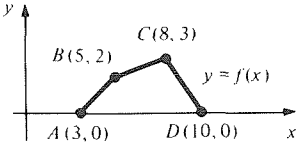

Figure 4.5.4. Find a formula for f.

41. Let f be continuous on the interval I and let a_1 and a_2 be in I. Define the functions:

$$F_1(t) = \int_{a_1}^t f(s)\,ds \quad \text{and} \quad F_2(t) = \int_{a_2}^t f(s)\,ds.$$

(a) Show that F_1 and F_2 differ by a constant.

(b) Express the constant $F_2 - F_1$ as an integral.

42. Develop a formula for $\int x(1+x)^n\,dx$ for $n \neq -1$ or -2 by studying Example 2 [*Hint*: Guess the answer $(ax + b)(1 + x)^{n+1}$ and determine what a and b have to be.]

★43. (a) Combine the alternative version of the fundamental theorem of calculus with the chain rule to prove that

$$\frac{d}{dt}\int_a^{g(t)} f(s)\,ds = f(g(t)) \cdot g'(t).$$

(b) Interpret (a) in terms of Fig. 4.5.5.

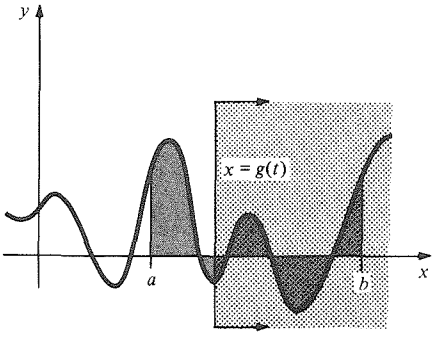

Figure 4.5.5. The rate of change of the exposed area with respect to t is $f(g(t)) \cdot g'(t)$.

★44. Compute $\frac{d}{dt}\int_0^{t^2} \frac{dx}{1+x^2}$.

★45. Let $F(x) = \int_1^{x^2} \frac{dt}{t}$. What is $F'(x)$?

★46. Calculate $\frac{d}{ds}\int_0^{s^4+s^2} \frac{dx}{1+x^4}$.

★47. Find a formula for $\frac{d}{dt}\int_{h(t)}^{g(t)} f(x)\,dx$ and explain your formula in terms of areas.

★48. Compute $\frac{d}{ds}\int_{s^2}^{s^3} \frac{x^3}{1+x^4}\,dx$.

Exercises 49–53 outline a proof of this fact: *if f is continuous on $[a, b]$, then $F(t) = \int_a^t f(s)\,ds$ is an antiderivative of f.*

★49. Prove property 3 for an integrable function f; that is, if f is integrable on $[a, b]$ and on $[b, c]$, then f is integrable on $[a, c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

[*Hint*: Let I be the right-hand side. Show that every number less than I is a lower sum for f on $[a, c]$ and, likewise, every number greater than I is an upper sum. If $S < I$, show by a general fact about inequalities that you can write $S = S_1 + S_2$, where $S_1 < \int_a^b f(x)\,dx$ and $S_2 < \int_b^c f(x)\,dx$. Now piece together a lower sum corresponding to S_1 with one for S_2.]

★50. Prove property 5 for integrable functions f and g. [*Hint*: Every lower sum for f is also one for g.]

★51. Show that

$$\frac{F(t+h) - F(t)}{h} = \frac{1}{h}\int_t^{t+h} f(s)\,ds$$

using property 3 of the integral.

★52. Show that $(1/h)\int_t^{t+h} f(s)\,ds$ lies between the maximum and minimum values of f on the interval $[t, t + h]$ (you may assume $h > 0$; a similar argument is needed for $h < 0$). Conclude that

$$\frac{1}{h}\int_t^{t+h} f(s)\,ds = f(c)$$

for some c between t and $t + h$, by the intermediate value theorem. (This result is sometimes called the *mean value theorem for integrals*; we will treat it again in Section 9.3.)

★53. Use continuity of f and the results from Exercises 51 and 52 to show that $F' = f$.

★54. Exercises 49–53 outlined a complete proof of the alternative version of the fundamental theorem of calculus. Use the "alternative version" to prove the "main version" in Section 4.4.

Assume that $F' = f$ is continuous.

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4.6 Applications of the Integral

Areas between graphs can be calculated as integrals.

We have seen that area under the graph of a function can be expressed as an integral. After a word problem based on this fact, we will learn how to calculate areas *between* graphs in the plane. Other applications of integration concern recovering the total change in a quantity from its rate of change.

Example 1 A parabolic doorway with base 6 feet and height 8 feet is cut out of a wall. How many square feet of wall space are removed?

Solution Place the coordinate system as shown in Fig. 4.6.1 and let the parabola have equation $y = ax^2 + c$. Since $y = 8$ when $x = 0$, $c = 8$. Also, $y = 0$ when $x = 3$, so $0 = a3^2 + 8$, so $a = -\frac{8}{9}$. Thus the parabola is $y = -\frac{8}{9}x^2 + 8$. The area under its graph is

$$\int_{-3}^{3}\left(-\frac{8}{9}x^2 + 8\right)dx = \left(-\frac{8}{27}x^3 + 8x\right)\Big|_{-3}^{3}$$
$$= \left(-\frac{8}{27}\cdot 27 + 8\cdot 3\right) - \left(\frac{8}{27}\cdot 27 - 8\cdot 3\right) = 32,$$

so 32 square feet have been cut out. ▲

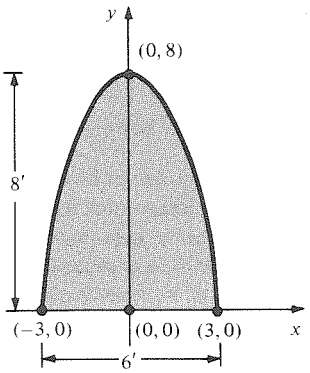

Figure 4.6.1. Find the area of the parabolic doorway.

Now we turn to the problem of finding the area between the graphs of two functions. If f and g are two functions defined on $[a, b]$, with $f(x) \leqslant g(x)$ for all x in $[a, b]$, we define *the region between the graphs* of f and g on $[a, b]$ to be the set of those points (x, y) such that $a \leqslant x \leqslant b$ and $f(x) \leqslant y \leqslant g(x)$.

Example 2 Sketch and find the area of the region between the graphs of x^2 and $x + 3$ on $[-1, 1]$.

Solution The region is shaded in Fig. 4.6.2. It is not quite of the form we have been dealing with; however, we may note that if we add to it the region under the graph of x^2 on $[-1, 1]$, we obtain the region under the graph of $x + 3$ on $[-1, 1]$. Denoting the area of the shaded region by A, we have

$$\int_{-1}^{1} x^2\,dx + A = \int_{-1}^{1}(x + 3)\,dx$$

or

$$A = \int_{-1}^{1}(x + 3)\,dx - \int_{-1}^{1} x^2\,dx = \int_{-1}^{1}(x + 3 - x^2)\,dx.$$

Evaluating the integral yields $A = (\frac{1}{2}x^2 + 3x - \frac{1}{3}x^3)\big|_{-1}^{1} = 5\frac{1}{3}$. ▲

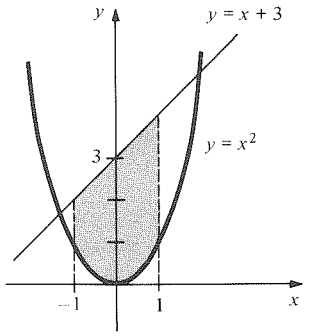

Figure 4.6.2. The area of the shaded region is the difference between two areas under graphs.

The method of Example 2 can be used to show that if $0 \leqslant f(x) \leqslant g(x)$ for x in $[a, b]$, then the area of the region between the graphs of f and g on $[a, b]$ is equal to $\int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b [g(x) - f(x)]\,dx$.

Example 3 Find the area between the graphs of $y = x^2$ and $y = x^3$ for x between 0 and 1.

Solution Since $0 \leqslant x^3 \leqslant x^2$ on $[0, 1]$, by the principle just stated the area is

$$\int_0^1 (x^2 - x^3)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}\cdot$$ ▲

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The same method works even if $f(x)$ can take negative values. It is only the *difference* between $f(x)$ and $g(x)$ which matters.

Area Between Graphs

If $f(x) \leqslant g(x)$ for all x in $[a,b]$, and f and g are integrable on $[a,b]$, then the area between the graphs of f and g on $[a,b]$ equals

$$\int_a^b [\, g(x) - f(x) \,]\, dx.$$

There is a heuristic argument for this formula for the area between graphs which gives a useful way of remembering the formula and deriving similar ones. We can think of the region between the graphs as being composed of infinitely many "infinitesimally wide" rectangles, of width dx, one for each x in $[a,b]$. (See Fig. 4.6.3.)

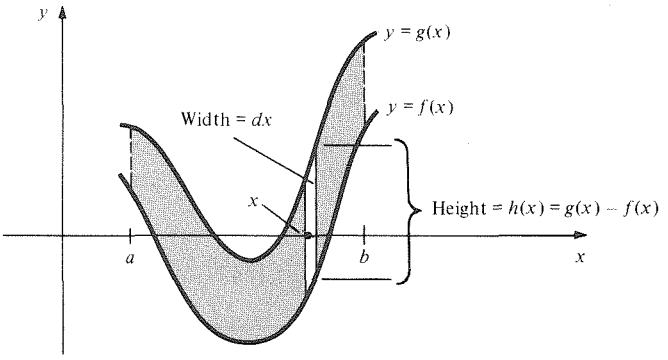

Figure 4.6.3. We may think of the shaded region as being composed of infinitely many rectangles, each of infinitesimal width.

The total area is then the "continuous sum" of the areas of these rectangles. The height of the rectangle over x is $h(x) = g(x) - f(x)$, the area of the rectangle is $[\, g(x) - f(x)]\, dx$, and the continuous sum of these areas is the integral $\int_a^b [\, g(x) - f(x)]\, dx$. This kind of infinitesimal argument was used frequently in the early days of calculus, when it was considered to be perfectly acceptable. Nowadays, we usually take the viewpoint of Archimedes, who used infinitesimals to discover results which he later proved by more rigorous, but much more tedious, arguments.

Example 4 Sketch and find the area of the region between the graphs of x and $x^2 + 1$ on $[-2, 2]$.

Solution The region is shaded in Fig. 4.6.4. By the formula for the area between curves, the area is

$$\int_{-2}^{2} [(x^2 + 1) - (x)]\, dx = \int_{-2}^{2} (x^2 + 1 - x)\, dx = \left(\frac{x^3}{3} + x - \frac{x^2}{2} \right) \Bigg|_{-2}^{2}$$

$$= \left(\frac{8}{3} + 2 - \frac{4}{2} \right) - \left(-\frac{8}{3} - 2 - \frac{4}{2} \right) = \frac{28}{3}. \blacktriangle$$

If the graphs of f and g intersect, then the area of the region between them must be found by breaking the region up into smaller pieces and applying the preceding method to each piece.

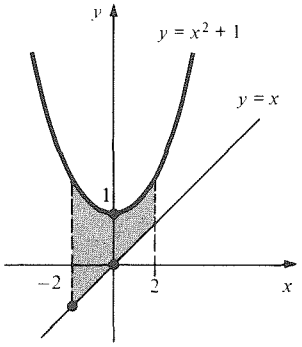

Figure 4.6.4. What is the area of the shaded region?

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Area Between Intersecting Graphs

To find the area of the region between the graphs of f and g and between $x = a$ and $x = b$, first plot the graphs and locate points where $f(x) = g(x)$. Suppose, for example, that $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant c$, $f(c) = g(c)$ and $f(x) \leqslant g(x)$ for $c \leqslant x \leqslant b$, as in Fig. 4.6.5. Then the area is

$$A = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx.$$

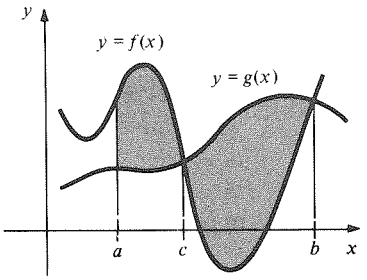

Figure 4.6.5. The area of the shaded region equals $\int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$.

Example 5 Find the area of the shaded region in Fig. 4.6.6.

Solution First we locate the intersection points by setting $x^2 = x$. This has the solutions $x = 0$ or 1. Between 0 and 1, $x^2 \leqslant x$, and between 1 and 2, $x^2 \geqslant x$, so the area is

$$A = \int_0^1 (x - x^2)\,dx + \int_1^2 (x^2 - x)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] = 1. \blacktriangle$$

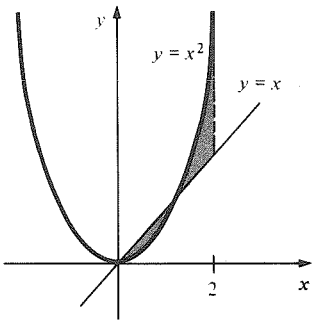

Figure 4.6.6. Find the shaded area.

Example 6 Find the area between the graphs of $y = x^3$ and $y = 3x^2 - 2x$ between $x = 0$ and $x = 2$.

Solution The graphs are plotted in Fig. 4.6.7. They intersect when $x^3 = 3x^2 - 2x$, i.e., $x(x^2 - 3x + 2) = 0$, i.e., $x(x - 2)(x - 1) = 0$, which has solutions $x = 0$, 1, and 2, as in the figure. The area is thus

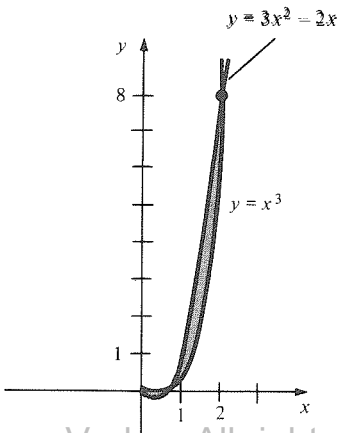

Figure 4.6.7. The graphs needed to find the area between $y = x^3$ and $y = 3x^2 - 2x$.

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$$\int_0^1 \left[x^3 - (3x^2 - 2x)\right] dx + \int_1^2 \left[(3x^2 - 2x) - x^3\right] dx$$

$$= \left[\frac{x^4}{4} - (x^3 - x^2)\right]\Big|_0^1 + \left[(x^3 - x^2) - \frac{x^4}{4}\right]\Big|_1^2$$

$$= \frac{1}{4}\left[(8 - 4) - \frac{16}{4} + \frac{1}{4}\right] = \frac{1}{2}. \blacktriangle$$

In the next problem, the intersection points of two graphs determine the limits of integration.

Example 7 The curves $x = y^2$ and $x = 1 + \frac{1}{2}y^2$ (neither of which is the graph of a function $y = f(x)$) divide the xy plane into five regions, only one of which is bounded. Sketch and find the area of this bounded region.

Solution If we plot x as a function of y, we obtain the graphs and region shown in Fig. 4.6.8. We use our general rule for the area between graphs, which gives

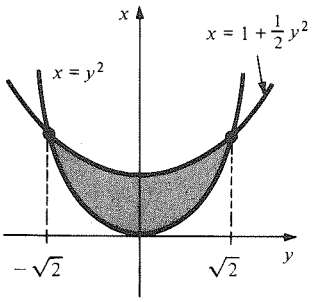

Figure 4.6.8. The bounded region determined by $x = y^2$ and $x = 1 + \frac{1}{2}y^2$.

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \left(1 + \frac{1}{2}y^2 - y^2\right) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(1 - \frac{1}{2}y^2\right) dy = \left(y - \frac{1}{6}y^3\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \sqrt{2} - \frac{1}{6}\left(\sqrt{2}\right)^3 - \left[-\sqrt{2} - \frac{1}{6}\left(-\sqrt{2}\right)^3\right]$$

$$= 2\sqrt{2} - \frac{1}{3}\left(\sqrt{2}\right)^3 = \sqrt{2}\left(2 - \frac{2}{3}\right) = \frac{4}{3}\sqrt{2} = \frac{2^{5/2}}{3}.$$

(Note that the roles of x and y have been reversed in this example.) $\blacktriangle$

Example 8 Find the area between the graphs $x = y^2 - 2$ and $y = x$.

Solution This is a good illustration of the fact that sometimes it is wise to pause and think about various methods at our disposal rather than simply plunging ahead.

First of all, we sketch the graphs, as in Fig. 4.6.9. We can plot $x = y^2 - 2$ either by writing $y = \pm\sqrt{x + 2}$ and graphing these two square-root functions or, preferably, by regarding x as a function of y and drawing the corresponding parabola.

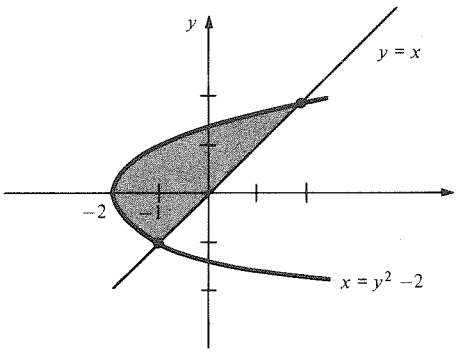

Figure 4.6.9. Find the shaded area.

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The intersection points are obtained by setting $y^2 - 2 = y$, which gives $y = -1$ and 2.

Method 1 Write $y = \pm\sqrt{x+2}$ for the first graph and obtain the area in two pieces as

$$A = \int_{-2}^{-1}\left[\sqrt{x+2} - \left(-\sqrt{x+2}\right)\right]dx + \int_{-1}^{2}\left[\sqrt{x+2} - x\right]dx$$

$$= \frac{4}{3}(x+2)^{3/2}\Big|_{-2}^{-1} + \left(\frac{2}{3}(x+2)^{3/2} - \frac{x^2}{2}\right)\Big|_{-1}^{2}$$

$$= \frac{4}{3} - 0 + \left(\frac{2}{3}\cdot 8 - 2\right) - \left(\frac{2}{3} - \frac{1}{2}\right) = \frac{9}{2}.$$

Method 2 Regard x as a function of y and obtain the area as

$$\int_{-1}^{2}\left[y - (y^2 - 2)\right]dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 2y\right)\Big|_{-1}^{2}$$

$$= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \frac{9}{2}.$$

Method 2, while requiring the slight trick of regarding x as a function of y, is simpler. ▲

The velocity–displacement relationship holds for all rates of change. If a quantity Q depends on x and has a rate of change r, then by the fundamental theorem,

$$Q(b) - Q(a) = \int_a^b r\,dx.$$

Total Changes from Rates of Change

If the rate of change of Q with respect to x for $a \leqslant x \leqslant b$ is given by $r = f(x)$, then the change in Q is obtained by integrating:

$$\Delta Q = Q(b) - Q(a) = \int_a^b f(x)\,dx.$$

This relationship can be used in a variety of ways, depending on the interpretation of Q, r, and x. For example, we can view the box on page 230 as a special instance with Q the position, r the velocity, and x the time.

Example 9 Water is flowing into a tub at $3t^2 + 6t$ liters per minute at time t, between $t = 0$ and $t = 2$. How many liters enter the tub during this period?

Solution Let $Q(t)$ denote the number of liters at time t. Then $Q'(t) = 3t^2 + 6t$, so

$$Q(2) - Q(0) = \int_0^2 (3t^2 + 6t)\,dt = (t^3 + 3t^2)\Big|_0^2 = 20.$$

Thus, 20 liters enter the tub during the 3-minute interval. ▲

The final example comes from economics. We recall from the discussion on page 106 that the marginal revenue at production level x is $R'(x)$, where R is the revenue.

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Example 10 The marginal revenue for a company at production level x is given by $15 - 0.1x$. If $R(x)$ denotes the revenue and $R(0) = 0$, find $R(100)$.

Solution By the fundamental theorem, $R(100) - R(0) = \int_0^{100} R'(x)\,dx$. But $R(0) = 0$, and the marginal revenue $R'(x)$ is $15 - 0.1x$. Thus,

$$R(100) = \int_0^{100} (15 - 0.1x)\,dx$$

$$= \left(15x - \frac{0.1x^2}{2}\right)\Bigg|_0^{100}$$

$$= 1500 - 500 = 1000. \blacktriangle$$

Exercises for Section 4.6

1. A parabolic arch with base 8 meters and height 10 meters is erected. How much area does it enclose?
2. A parabolic arch with base 10 meters and height 12 meters is erected. How much area does it enclose?
3. A swimming pool has the shape of the region bounded by $y = x^2$ and $y = 2$. A swimming pool cover is estimated to cost \$2.00 per square foot. If one unit along each of the x and y axes is 50 feet, then how much should the cover cost?
4. An artificial lake with two bays has the shape of the region above the curve $y = x^4 - x^2$ and below the line $y = 8$ (x and y are measured in kilometers). If the lake is 10 meters deep, how many cubic meters of water does it hold?
5. Find the area of the shaded region in Fig. 4.6.10.

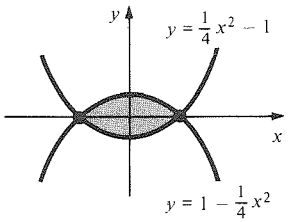

Figure 4.6.10. What is the area of the shaded region?

6. Find the area of the shaded region in Fig. 4.6.11.

Find the area between the graphs on the designated intervals in Exercises 7–10.

7. $y = (2/x^2) + x^4$ and $y = 1$ between $x = 1$ and $x = 2$.
8. $y = x^4$ and $y = x^3$ between $x = -1$ and $x = 0$.
9. $y = \sqrt{x}$ and $y = x$ between $x = 0$ and $x = 1$.
10. $y = \sqrt[3]{x}$ and $y = 1/x^2$ between $x = 8$ and $x = 27$.

In Exercises 11–16, find the area between the graphs of each pair of functions on the given interval:

11. x and x^4 on $[0, 1]$.
12. x^2 and $4x^4$ on $[2, 3]$.
13. $3x^2$ and $x^4 + 2$ on $[-\frac{1}{2}, \frac{1}{2}]$.
14. $x^4 + 1$ and $1/x^2$ on $[1, 2]$.
15. $3 + \dfrac{x^4 + x^2}{x^{59} + x^4}$ and $7 + \dfrac{x^4 + x^2}{x^{59} + x^4}$ on $[46917, 46919]$.
16. $\dfrac{4(x^6 - 1)}{x^6 + 1}$ and $\dfrac{(3x^6 - 1)(x^6 - 1)}{x^6(x^6 + 1)}$ on $[1, 2]$.
17. Find the area between the graphs of $y = x^3$ and $y = 5x^2 + 6x$ between $x = 0$ and $x = 3$.
18. Find the area between the graphs of $y = x^3 + 1$ and $y = x^2 - 1$ between $x = -1$ and $x = 1$.
19. The curves $y = x^3$ and $y = x$ divide the plane into six regions, only two of which are bounded. Find the areas of the bounded regions.
20. The lines $y = x$ and $y = 2x$ and the curve $y = 2/x^2$ together divide the plane into several regions, one of which is bounded. (a) How many regions are there? (b) Find the area of the bounded region.

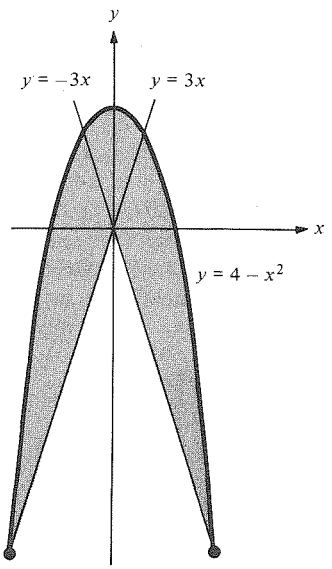

Figure 4.6.11. What is the area of the shaded region?

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In Exercises 21–24, find the area between the given graphs.

21. $x = y^2 - 3$ and $x = 2y$
22. $x = y^2 + 8$ and $x = -6y$
23. $x = y^3$ and $y = 2x$
24. $x = y^4 - 2$ and $x = y^2$
25. Water is flowing out of a reservoir at $300t^2$ liters per second for t between 0 and 5. How many liters are released in this period?
26. Air is escaping from a balloon at $3t^2 + 2t$ cubic centimeters per second for t between 1 and 3. How much air escapes during this period?
27. Suppose that the marginal revenue of a company at production level x is given by $30 - 0.02x - 0.0001x^2$. If $R(0) = 0$, find $R(300)$.
28. (a) Find the total revenue $R(x)$ from selling x units of a product, if the marginal revenue is $36 - 0.01x + 0.00015x^2$ and $R(0) = 0$.
    (b) How much revenue is produced when the items from $x = 50$ to $x = 100$ are sold?
29. (a) Use calculus to find the area of the triangle whose vertices are $(0,0)$, (a,h), and $(b,0)$. (Assume $0 < a < b$ and $0 < h$.) Compare your result with a formula from geometry.
    (b) Repeat part (a) for the case $0 < b < a$.
30. *In Example 2 of Section 4.5, it was shown that* $\int x(1+x)^6\,dx = \frac{1}{56}(7x-1)(1+x)^7 + C$. Use this result to find the area under the graph of $1 + x(1+x)^6$ between $x = -1$ and $x = 1$.
31. Fill in the blank, referring to the Supplement to Section 4.3: Light meter is to energy-storage dial as _____ *is to* odometer.
32. Fill in the blank: Marginal revenue is to _____ as $f(x)$ is to $\int_a^x f(s)\,ds$.
33. A circus tent is equipped with four exhaust fans at one end, each capable of moving 5500 cubic feet of air per minute. The rectangular base of the tent is 80 feet by 180 feet. Each corner is supported by a 20-foot-high post and the roof is supported by a center beam which is 32 feet off the ground and runs down the center of the tent for its 180-foot length. Canvas drapes in a parabolic shape from the center beam to the sides 20 feet off the ground. (See Fig. 4.6.12.) Determine the elapsed time for a complete change of air in the tent enclosure.

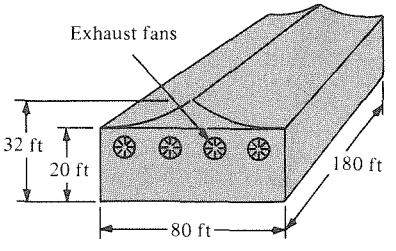

Figure 4.6.12. The circus tent in Exercise 33.

34. A small gold mine in northern Nevada was reopened in January 1979, producing 500,000 tons of ore in the first year. Let $A(t)$ be the number of tons of ore produced, in thousands, t years after 1979. Productivity $A'(t)$ is expected to decline by 20,000 tons per year until 1990.
    (a) Find a formula for $A'(t)$, assuming that the production decline is constant.
    (b) How much ore is mined, approximately, T years after 1979 during a time period Δt?
    (c) Find, by definite integration, the predicted number of tons of ore to be mined from 1981 through 1986.
35. Let $W(t)$ be the number of words learned after t minutes are spent memorizing a French vocabulary list. Typically, $W(0) = 0$ and $W'(t) = 4(t/100) - 3(t/100)^2$.
    (a) Apply the fundamental theorem of calculus to show that
    $$W(t) = \int_0^t \left[4(x/100) - 3(x/100)^2\right] dx.$$
    (b) Evaluate the integral in part (a).
    (c) How many words are learned after 1 hour and 40 minutes of study?
36. The region under the graph $y = 1/x^2$ on $[1,4]$ is to be divided into two parts of equal area by a vertical line. Where should the line be drawn?

★37. Where would you draw a horizontal line to divide the region in the preceding exercise into two parts of equal area?

★38. Find the area in square centimeters, correct to within 1 square centimeter, of the region in Fig. 4.6.13.

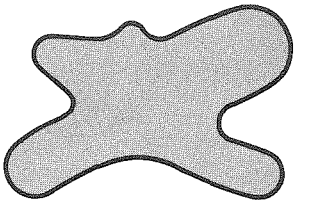

Figure 4.6.13. Find the area of the "blob."

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Review Exercises for Chapter 4

Compute the sums in Exercises 1–8.

1. $\sum_{i=1}^{4} i^2$.
2. $\sum_{j=1}^{3} j^3$.
3. $\sum_{i=1}^{5} \frac{2^i}{i(i+1)}$.
4. $\sum_{j=4}^{8} \frac{j^2-10}{3j}$.
5. $\sum_{i=1}^{500} (3i+7)$.
6. $\sum_{i=n}^{n+3} \frac{i^2-1}{i+1}$ (n is a non-negative integer).
7. $\sum_{i=0}^{10} [(i+1)^4 - i^4]$.
8. $\sum_{i=2}^{60} \left(\frac{1}{i} - \frac{1}{i-1}\right)$.
9. Let f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 \leqslant x < \frac{1}{5}, \\ 2, & \frac{1}{5} \leqslant x < \frac{1}{4}, \\ 3, & \frac{1}{4} \leqslant x < \frac{1}{3}, \\ 4, & \frac{1}{3} \leqslant x < \frac{1}{2}, \\ 5, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

   Find $\int_0^1 f(x)\,dx$.

10. Let f be defined by

$$f(x) = \begin{cases} -1, & -1 \leqslant x < 0, \\ 2, & 0 \leqslant x < 1, \\ 3, & 1 \leqslant x \leqslant 2. \end{cases}$$

    Find $\int_{-1}^2 f(x)\,dx$.

11. Interpret the integral in Exercise 9 in terms of distances and velocities.
12. Interpret the integral in Exercise 10 in terms of distances and velocities.

Evaluate the definite integrals in Exercises 13–16.

13. $\int_3^5 (-2x^3 + x^2)\,dx$.
14. $\int_1^3 \frac{x^3-5}{x^2}\,dx$.
15. $\int_1^2 \frac{\frac{1}{3}s^2 - (s^4+1)}{2s^2}\,ds$.
16. $\int_1^2 \frac{x^2+3x+2}{x+1}\,dx$.

Find the area under the graphs of the functions between the indicated limits in Exercises 17–20.

17. $y = x^3 + x^2$, $0 \leqslant x \leqslant 1$.
18. $y = \frac{x^2+2x+1}{x^4}$, $1 \leqslant x \leqslant 2$.
19. $y = (x+3)^{4/3}$, $0 \leqslant x \leqslant 2$.
20. $y = (x-1)^{1/2}$, $1 \leqslant x \leqslant 2$.

21. (a) Find upper and lower sums for

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

within 0.2 of one another. (b) Look at the average of these sums. Can you guess what the exact integral is?

22. Find upper and lower sums for $\int_2^3 \frac{1}{x}\,dx$ within $\frac{1}{10}$ of one another.
23. (a) Find $\frac{d}{dx}\left[\frac{1}{(1+x^2)}\right]$.
    (b) Find the area under the graph of the function $x/(1+x^2)^2$, from $x = 0$ to $x = 1$.
24. Find $(d/dx)[x^3/(1+x^3)]$. (b) Find the area under the graph of $x^2/(1+x^3)^2$ from $x = 1$ to $x = 2$.
25. Find the area under the graph of $y = mx + b$ from $x = a_1$ to $x = a_2$ and verify your answer by using plane geometry. Assume that $mx + b \geqslant 0$ on $[a_1, a_2]$.
26. Find the area under the graph of the function $y = (1/x^2) + x + 1$ from $x = 1$ to $x = 2$.
27. Find the area under the graph of $y = x^2 + 1$ from -1 to 2 and sketch the region.
28. Find the area under the graph of

$$f(x) = \begin{cases} -x^3 & \text{if } x \leqslant 0, \\ x^3 & \text{if } x > 0 \end{cases}$$

from $x = -1$ to $x = 1$ and sketch.

29. (a) Verify the integration formula

$$\int \frac{x^2}{(x^3+6)^2}\,dx = \frac{1}{12}\left[\frac{x^3+2}{(x^3+6)}\right] + C.$$

    (b) Find the area under the graph of $y = x^2/(x^3+6)^2$ between $x = 0$ and $x = 2$.
30. Find the area between the graphs of $y = x^3$ and $y = 5x^2 + 2x$ between $x = 0$ and $x = 2$. Sketch.
31. The curves $y = x^6 - 3$ and $y = -x^2 - 1$ divide the plane into five regions, one of which is bounded. Find its area.
32. Find the area of each of the numbered regions in Fig. 4.R.1.

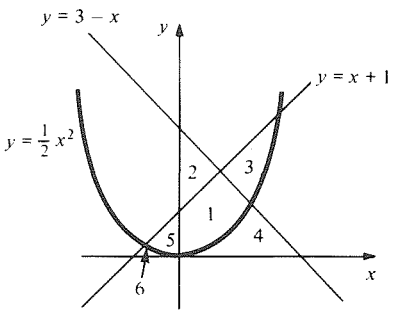

Figure 4.R.1. Find the area of the numbered regions.

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33. Find the area of the region bounded by the graphs $x = y^2 - 6$ and $x = y$.
34. Find the area of the region bounded by the graphs $y = x^2 - 2$ and $y = 2 - x^2$.
35. An object is thrown at $t = 0$ from an airplane, and it has vertical velocity $v = -10 - 32t$ feet per second at time t. If the object is still falling after 10 seconds, what can you say about the altitude of the plane at $t = 0$?
36. Suppose the velocity of an object at position x is x^n, where n is some integer $\neq 1$. Find the time required to travel from $x = \frac{1}{1000}$ to $x = 1$.
37. (a) At time $t = 0$, a container has 1 liter of water in it. Water is poured in at the rate of $3t^2 - 2t + 3$ liters per minute (t = time in minutes). If the container has a leak which can drain 2 liters per minute, how much water is in the container at the end of 3 minutes?
    (b) What if the leak is 4 liters per minute?
    (c) What if the leak is 8 liters per minute? [*Hint*: What happens if the tank is empty for a while?]
38. Water is poured into a container at a rate of t liters per minute. At the same time, water is leaking out at the rate of t^2 liters per minute. Assume that the container is empty at $t = 0$.
    (a) When does the amount of water in the container reach its maximum?
    (b) When is the container empty again?
39. Suppose that a *supply curve* $p = S(x)$ and a *demand curve* $p = D(x)$ are graphed and that there is a unique point (a, b) at which supply equals demand (p = price/unit in dollars, x = number of units). The (signed) area enclosed by $x = 0$, $x = a$, $p = b$, and $p = D(x)$ is called the *consumer's surplus* or the *consumer's loss* depending on whether the sign is positive or negative. Similarly, the (signed) area enclosed by $x = 0$, $x = a$, $p = b$, and $p = S(x)$ is called the *producer's surplus* or the *producer's loss* depending on whether the sign is positive or negative.
    (a) Let $D(x) \geqslant b$. Explain why the *consumer's surplus is* $\int_0^a [D(x) - b]\,dx$.
    (b) Let $S(x) \leqslant b$. Explain why the *producer's surplus is* $\int_0^a [b - S(x)]\,dx$.
    (c) "If the price stabilizes at \$6 per unit, then some people are still willing to pay a higher price, but benefit by paying the lower price of \$6 per unit. The total of these benefits over $[0, a]$ is the consumer's surplus." Explain this in the language of integration.
    (d) Find the consumer's and producer's surplus for the supply curve $p = x^2/8$ and the demand curve $p = -(x/4) + 1$.
40. The demand for wood products in 1975 was about 12.6 billion cubic feet. By measuring order increases, it was determined that x years after 1975, the demand increased by $9x/1000$; that is, $D'(x) = 9x/1000$, where $D(x)$ is the demand x years after 1975, in billions of cubic feet.
    (a) Use the fundamental theorem of calculus to show that
    $$D(x) = D(0) + \int_0^x (9t/1000)\,dt.$$
    (b) Find $D(x)$.
    (c) Find the demand for wood in 1982.
41. Suppose that an object on the x axis has velocity $v = t^2 - 4t - 5$. How far does it travel between $t = 0$ and $t = 6$?
42. Show that the actual distance travelled by a bus with velocity $v = f(t)$ is $\int_a^b |f(t)|\,dt$. What condition on $v = f(t)$ means that the bus made a round trip between $t = a$ and $t = b$?
43. A rock is dropped off a bridge over a gorge. The sound of the splash is heard 5.6 seconds after the rock was dropped. (Assume the rock falls with velocity $32t$ feet per second and sound travels at 1080 feet per second.)
    (a) Show by integration that the rock falls $16t^2$ feet after t seconds, and that the sound of the splash travels $1080t$ feet in t seconds.
    (b) The time T required for the rock to hit the water must satisfy $16T^2 = 1080(5.6 - T)$, because the rock and the sound wave travel equal distances. Find T.
    (c) Find the height of the bridge.
    (d) Find the number of seconds required for the sound of the splash to travel from the water to the bridge.
44. The current $I(t)$ and charge $Q(t)$ at time t (in amperes and coulombs, respectively) in a circuit are related by the equation $I(t) = Q'(t)$.
    (a) Given $Q(0) = 1$, use the fundamental theorem of calculus to justify the formula $Q(t) = 1 + \int_0^t I(r)\,dr$.
    (b) The voltage drop V (in volts) across a resistor of resistance R ohms is related to the current I (in amperes) by the formula $V = RI$. Suppose that in a simple circuit with a resistor made of nichrome wire, $V = 4.36$, $R = 1$, and $Q(0) = 1$. Find $Q(t)$.
    (c) Repeat (b) for a circuit with a 12-volt battery and 4-ohm resistance.
45. A ruptured sewer line causes lake contamination near a ski resort. The concentration $C(t)$ of bacteria (number per cubic centimeter) after t days is given by $C'(t) = 10^3(t - 7)$, $0 \leqslant t \leqslant 6$, after treatment of the lake at $t = 0$.
    (a) An inspector will be sent out after the bacteria concentration has dropped to half its original value $C(0)$. On which day should the inspector be sent if $C(0) = 40{,}000$?
    (b) What is the total change in the concentration from the fourth day to the sixth day?

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46. A baseball is thrown vertically upwards from the ground with an initial upward velocity of 50 feet per second. How far has it travelled when it strikes the ground?

47. Let

$$g(y) = \begin{cases} y, & 0 \leq y < 1, \\ 2, & 1 \leq y < 2, \\ y, & 2 \leq y \leq 4. \end{cases}$$

Compute $\int_0^1 g(y)\,dy + \int_3^4 g(y)\,dy$.

48. Let

$$y(t) = \begin{cases} t, & 2 \leq t < 3, \\ -4, & 3 \leq t < 4, \\ 1, & 4 \leq t \leq 5. \end{cases}$$

Compute $\int_5^2 y(t)\,dt$.

49. Calculate $\dfrac{d}{dx}\int_0^x \dfrac{s^2}{1+s^3}\,ds$.

50. Calculate $\dfrac{d}{dx}\int_2^x \dfrac{t^3}{1-t^4}\,dt$.

51. Find the area between the graph of the function in Exercise 47 and the x axis.

★52. Find $\dfrac{d}{ds}\int_0^{s^2} \dfrac{1}{\sqrt[3]{x^2+1}}\,dx$. (*Hint*: See Exercise 43, Section 4.5.)

★53. Find $\dfrac{d}{dt}\int_0^{t^3+2} \dfrac{1}{y^2+1}\,dy$.

★54. Find $\int_a^b (3x^2 + x)\,dx$ "by hand."

★55. Let f be defined on [0, 1] by

$$f(t) = \begin{cases} 0, & x = 0, \\ 1/\sqrt{x}, & 0 < x \leq 1. \end{cases}$$

(a) Show that there are no upper sums for f on [0, 1], and hence that f is not integrable.

(b) Show that every number less than 2 is a lower sum. [*Hint*: Use step functions which are zero on an interval $[0, \epsilon)$ and approximate f very closely on $[\epsilon, 1]$. Take ϵ small and use the integrability of f on $[\epsilon, 1]$.]

(c) Show that no number greater than or equal to 2 is a lower sum. [*Hint*: Show $\epsilon f(\epsilon) + \int_\epsilon^1 f(x)\,dx < 2$ for all ϵ in (0, 1).]

(d) If you had to assign a value to $\int_0^1 f(x)\,dx$, what value would you assign?

★56. Modeling your discussion after the preceding exercise, find the upper or lower sums for each of the following functions on [0, 1]:

(a) $f(x) = \begin{cases} 0, & x = 0, \\ -1/\sqrt[3]{x}, & x > 0; \end{cases}$

(b) $f(x) = \begin{cases} 0, & x = 0, \\ \dfrac{1}{x^2}, & x > 0. \end{cases}$

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